# COMPLETELY DECOMPOSABLE JACOBIAN VARIETIES IN NEW GENERA

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Abstract. We present a new technique to study Jacobian variety decompositions using subgroups of the automorphism group of the curve and the corresponding intermediate covers. In particular, this new method allows us to produce many new examples of genera for which there is a curve with completely decomposable Jacobian. These examples greatly extend the list given by Ekedahl and Serre of genera containing such curves, and provide more evidence for a positive answer to two questions they asked. Additionally, we produce new examples of families of curves, all of which have completely decomposable Jacobian varieties. These families relate to questions about special subvarieties in the moduli space of principally polarized abelian varieties.

## 1. Introduction

A principally polarized abelian variety over $\mathbb{C}$ is called *completely decomposable* if it is isogenous to a product of elliptic curves. In [Ekedahl and Serre 93] the following two questions are asked.

**Question 1.** Is it true that, for all positive integers $g$, there exists a curve of genus $g$ whose Jacobian is completely decomposable?

**Question 2.** Is the set of genera for which a curve with completely decomposable Jacobian exists infinite?

They demonstrate various curves up to genus 1297 with completely decomposable Jacobian varieties. However, there are numerous genera in that range for which they do not produce an example of a curve with this property.

Since their paper, there has been much interest in curves with completely decomposable Jacobian varieties, particularly the applications of such curves to number theory. Dimension two has been widely studied; for example, in [Earle 06] a full classification of Riemann matrices of *strictly* completely decomposable Jacobian varieties of dimension 2 is given (these are Jacobians which are *isomorphic* to a product of elliptic curves). In [Kani 94], the case of completely decomposable abelian surfaces is studied, and several other authors have also studied these questions. See [Carocca et al. 14], [Magaard et al. 09], [Nakajima 07], and [Yamauchi 07], among many others.

Additionally, in [Moonen and Oort 11, Question 6.6] the authors ask about positive-dimensional special subvarieties, $Z$, of the closure of the Jacobian locus in the

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moduli space of principally polarized abelian varieties such that the abelian variety corresponding with the geometric generic point of $Z$ is isogenous to a product of elliptic curves. In Section 3.3, we discuss examples of positive-dimensional families of curves with completely decomposable Jacobians, and connections to this question.

Despite advancements in the field, the questions of [Ekedahl and Serre 93] still remain open. Since the publication of Ekedahl and Serre's list 20 years ago, there have been few new examples of curves with completely decomposable Jacobians in a genus not included on that list. In [Yamauchi 07], the author gives a list of integers $N$ such that the Jacobian variety $J_0(N)$ of the modular curve $X_0(N)$ has elliptic curves as $\mathbb{Q}-$simple factors. These examples include three genera not previously noted in [Ekedahl and Serre 93] for which there is a completely decomposable Jacobian variety: these are genus $113, 161$, and $205$ (corresponding to $N = 672, 1152$, and $1200$, respectively). His techniques are number theoretic and relate to [Ekedahl and Serre 93, Section 2].

In this paper, we use experimental tools to find many examples of completely decomposable Jacobian varieties in new genera. To find these examples, we use the action of the automorphism groups on curves, particularly a new approach involving known results on *intermediate coverings*, i.e., quotients by the action of subgroups of the full group acting on the variety.

We summarize the main results of this work in the following theorem. The bold numbers indicate genera which are new in this paper.

**Theorem.** For every $g \in$ {1–29, **30**, 31, **32**, 33, **34–36**, 37, **39**, 40, 41, **42**, 43, **44**, 45, **46**, 47, **48**, 49, 50, **51–52**, 53, **54**, 55, 57, **58**, 61, **62–64**, 65, **67**, **69**, **71–72**, 73, **79–81**, 82, **85**, **89**, **91**, **93**, **95**, 97, **103**, **105–107**, 109, **118**, 121, **125**, 129, **142**, 145, **154**, 161, 163, **193**, **199**, **211**, **213**, 217, **244**, 257, 325, 433} there is a curve of genus $g$ with completely decomposable Jacobian variety found using a group acting on a curve.

In some cases there is a family of dimension greater than 0 with such a decomposition on the whole family. The Theorem includes all genera previously determined except for $g = 113, 205, 649$, and $1297$. Even for the already known genera, most of our examples are not the same as those found previously. Many of the examples found in [Ekedahl and Serre 93] use the theory of modular curves. We compared the automorphism groups of the modular curves given in [Ekedahl and Serre 93] to the automorphism groups in our examples and they are not equal. Genus 3 to 10, except genus 8 are in [Paulhus 08]. Genus 8 may be found with a curve of automorphism group of size 336 whose Jacobian is isogenous to $E^8$ for some elliptic curve $E$.

The previous theorem, and the approach we outline in Section 2.2, support the possibility that Question 1 has a positive answer, and that group actions might be the tool to answer it. As we will see, once there is a completely decomposable Jacobian variety in one larger genus, by considering subgroups it is possible to also produce new examples in lower genera. This provides a way to fill in gaps in the data.

We describe the techniques used to decompose Jacobians in Sections 2.1 and 2.2. In Section 3 we give explicit examples in both new and old genera. Our new examples may be found in Theorem 3.1 and Theorem 3.2. Those genera with a family of curves of dimension greater than 0 with completely decomposable Jacobians are given in Theorem 3.3. The computations needed to find both the old and new examples were made using Magma [Bosma et al. 97] and code to verify the decompositions is available at [Paulhus and Rojas 16]. Finally, we address computational limitations of our techniques in Section 4. The many examples from the paper may be useful to researchers interested in open questions surrounding curves with completely decomposable Jacobians.

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## 2. Techniques

Consider a compact Riemann surface $X$ (referred to from now on as a "curve") of genus $g$ with a finite group $G$ acting on that curve. We write the quotient curve $X/G$ as $X_G$ and the genus of the quotient as $g_0$. Let the cover $X \to X_G$ be branched over $r$ places, $q_1, \ldots, q_r \in X_G$. The signature of the cover is the $(r+1)$-tuple $[g_0; s_1, s_2, \ldots, s_r]$ where the $s_i$ are the ramification indices of the covering at the branch points. We denote the Jacobian variety of $X$ by $JX$.

### 2.1. The group algebra decomposition.

To find many examples, we use the group action of the automorphism group $G$ of $X$ to decompose $JX$. We briefly describe the technique here for a general abelian variety $A$. More details may be found in the original article [Lange and Recillas 04] or [Birkenhake and Lange 04, Chp. 13].

Let $A$ be an abelian variety of dimension $g$ with a faithful action of a finite group $G$. There is an induced homomorphism of $\mathbb{Q}$-algebras

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A).$$

Any element $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety

$$\alpha(A) := \mathrm{Im}(m\rho(\alpha)) \subset A$$

where $m$ is some positive integer such that $m\rho(\alpha) \in \mathrm{End}(A)$. This definition does not depend on the chosen integer $m$.

Begin with the decomposition of $\mathbb{Q}[G]$ as a product of simple $\mathbb{Q}$-algebras $Q_i$

$$\mathbb{Q}[G] = Q_1 \times \cdots \times Q_r.$$

The factors $Q_i$ correspond canonically to the rational irreducible representations $W_i$ of the group $G$, because each one is generated by a unit element $e_i \in Q_i$ which may be considered as a central idempotent of $\mathbb{Q}[G]$.

The corresponding decomposition of $1 \in \mathbb{Q}[G]$,

$$1 = e_1 + \cdots + e_r$$

induces an isogeny, via $\rho$ above,

$$e_1(A) \times \cdots \times e_r(A) \to A \tag{1}$$

which is given by addition. Note that the components $e_i(A)$ are $G$-stable complex subtori of $A$ with $\mathrm{Hom}_G(e_i(A), e_j(A)) = 0$ for $i \neq j$. The decomposition (1) is called the *isotypical decomposition* of the complex $G$-abelian variety $A$.

The isotypical components $e_i(A)$ can be decomposed further, using the decomposition of $Q_i$ into a product of minimal left ideals. If $W_i$ is the irreducible rational representation of $G$ corresponding to $e_i$ for every $i = 1, \dots, r$, and $\chi_i$ is the character of $U_i$, one of the irreducible $\mathbb{C}$-representations associated to $W_i$, then set

$$n_i = \frac{\dim U_i}{m_i}$$

where $m_i$ denotes the Schur index of $\chi_i$. There is a set of primitive idempotents $\{\pi_{i1}, \cdots, \pi_{in_i}\}$ in $Q_i \subset \mathbb{Q}[G]$ such that

$$e_i = \pi_{i1} + \cdots + \pi_{in_i}.$$

Moreover, the abelian subvarieties $\pi_{ij}(A)$ are mutually isogenous for fixed $i$ and $j = 1, \dots, n_i$. Call any one of these isogenous factors $B_i$. Then (see [Carocca and Rodríguez 06])

$$B_i^{n_i} \to e_i(A)$$

is an isogeny for every $i = 1, \dots, r$. Replacing the factors in (1) we get an isogeny called the *group algebra decomposition* of the $G$-abelian variety $A$

$$B_1^{n_1} \times \cdots \times B_r^{n_r} \to A. \tag{2}$$

Note that, whereas (1) is uniquely determined, (2) is not. It depends on the choice of the $\pi_{ij}$ as well as the choice of the $B_i$. However, the dimension of the factors will remain fixed regardless of these choices.

**Remark 2.1.** While the factors in (2) are not necessarily easy to determine, we may compute their dimension in the case of a Jacobian variety $JX$ with the action of a group $G$ induced by the action on the corresponding Riemann surface $X$ (see [Paulhus 08] for details). Define $V$ to be the representation of $G$ on $H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. As mentioned at the beginning of this section, here we assume the quotient $X_G$ has genus $g_0$ and the cover $\pi : X \to X_G$ has $r$ branch points $\{q_1 \dots, q_r\}$ where each $q_i$ has corresponding monodromy $c_i \in G$. The tuple $(c_1, \dots, c_r)$ is called the generating vector for the action [Broughton 90].

Then the character $\chi_V$ associated to $V$ is

$$\chi_V = 2\chi_{\mathrm{triv}} + 2\,(g_0 - 1)\,\rho_{\langle 1_G \rangle} + \sum_{i=1}^{r} \left(\rho_{\langle 1_G \rangle} - \rho_{\langle c_i \rangle}\right) \tag{3}$$

[Broughton 90, Equation 2.14], where $\chi_{\mathrm{triv}}$ is the trivial character on $G$, $\rho_H$ is the induced character on $G$ of the trivial character of the subgroup $H$ (when $H = \langle c_i \rangle$, this subgroup is the stabilizer, or isotropy group, of a point in the fiber of the branch point $q_i$), and $\rho_{\langle 1_G \rangle}$ is the character of the regular representation.

According to [Lange and Rojas 12, Eq. (3.4)], the dimension of a subvariety $B_i$ corresponding to the isotypical factor (2) associated to the (non-trivial) rational representation $W_i$ is

$$\dim_{\mathbb{C}} B_i = \frac{m_i[K_i:\mathbb{Q}]\langle\chi_V,\chi_i\rangle}{2} = \frac{\langle\chi_V, m_i[K_i:\mathbb{Q}]\chi_i\rangle}{2}, \tag{4}$$

where $\chi_i$ is the character of one of the irreducible $\mathbb{C}$-representations associated to $W_i$; this is, the character of a complex irreducible representation decomposing $W_i \otimes \mathbb{C}$, and $K_i$ is the field extension of $\mathbb{Q}$ containing all values of $\chi_i$ on elements of $G$.

It is a classical result in representation theory, see Proposition 2.2 below, that

$$W_i \otimes \mathbb{C} = m_i \bigoplus_{\sigma\in K_i} U_i^\sigma,$$

where $U_i$ is the complex irreducible representation affording $\chi_i$. Combining this decomposition with (4), we get

$$\dim_{\mathbb{C}} B_i = \frac{1}{2}\langle\chi_V,\psi_i\rangle \tag{5}$$

where here $\psi_i$ is the character of the $\mathbb{Q}$-irreducible representation $W_i$ of $G$ corresponding to $B_i$, and $\chi_V$ is the character defined in (3).

One way we find completely decomposable Jacobian varieties is to search for curves so that the decomposition in (2) gives factors $B_i$ of dimension only 0 or 1, computed via (5).

In practice, given a group $G$, we can use Magma to compute its $\mathbb{C}$-character table. Then we determine the characters of the irreducible $\mathbb{Q}$-representation using the following result.

**Proposition 2.2.** [Curtis and Reiner 62, Exercise 70.30.2] *Let $\{\chi_1,\ldots,\chi_r\}$ be the irreducible $\mathbb{C}$-characters of a finite group $G$. Then $\phi$ is an irreducible $\mathbb{Q}$-character if and only if $\phi = m_i\cdot(\chi_i+\chi_i^\sigma+\cdots)$ where the $\{\chi_i^\sigma\}$ are the distinct conjugates of $\chi_i$, an irreducible $\mathbb{C}$-character of $G$.*

2.2. **Intermediate Covering Decomposition.** While the technique in the previous section gives us new examples of completely decomposable Jacobians in new genera (see Theorem 3.1), we can extend the technique by studying decompositions of intermediate coverings of a higher genus curve which has a known decomposition of its corresponding Jacobian variety. This idea expands the range of genera with completely decomposable Jacobians which can be found using group actions. We find many more new genera, as listed in Theorem 3.2.

To describe the technique, we begin with the following proposition.

**Proposition 2.3.** [Carocca and Rodríguez 06, Proposition 5.2] *Given a Galois cover $X \to X_G$, consider the group algebra decomposition* (2)

$$JX \sim B_1^{\frac{\dim V_1}{m_1}} \times\cdots\times B_r^{\frac{dim V_r}{m_r}}$$

*where $V_j$ is a complex irreducible representation associated to $B_j$. If $H$ is a subgroup of $G$ then the group algebra decomposition of $JX_H$ is given as*

$$JX_H \sim B_1^{\frac{\dim V_1^H}{m_1}} \times\cdots\times B_r^{\frac{\dim V_r^H}{m_r}} \tag{6}$$

*where $V_j^H$ is the subspace of $V_j$ fixed by $H$.*

By Frobenius Reciprocity, we know that

$$\dim {V_j}^H = \langle V_j, \rho_H \rangle \tag{7}$$

where $\langle V_j, \rho_H \rangle$ is the inner product of the characters of these representations. Suppose $X$ is a curve with a known Jacobian decomposition as in (2), not necessarily completely decomposable. Then apply the previous proposition to get a decomposition of $JX_H$ as in (6) where $JX_H$ will be completely decomposable precisely when $\langle V_j, \rho_H \rangle = 0$ for all $j$ such that $\dim B_j > 1$ for the $B_i$ in the decomposition of $JX$. We have thus proven:

**Proposition 2.4.** *Given the conditions in the previous proposition, assume that $\langle V_j, \rho_H \rangle = 0$ for all $j$ such that $\dim B_j > 1$. Then the Jacobian variety of the curve $X_H$ is completely decomposable.*

Notice that even though a Jacobian variety $JX$ may not be completely decomposable, a Jacobian $JX_H$ of some intermediate cover $X_H = X/H$ could decompose completely. This gives us a much richer set of curves to search through to find completely decomposable Jacobian varieties. There are numerous examples of curves in high genus whose Jacobians decompose into many elliptic curves, but may not be themselves completely decomposable. By applying Proposition 2.4, quotients of these curves may then be completely decomposable.

Let us demonstrate with a couple of examples. More details and several other examples may be found in Section 3.2. First, a note on our notation for the rest of the paper. In most instances, we will write a specific group as an ordered pair, where the first number is the order of the group and the second number is its number in the Magma or GAP database of groups of small order. Some of the Magma code we use requires all groups to be represented as permutation groups. Thus, throughout the rest of the paper, the numbers used to label a specific subgroup or conjugacy class of a group will be for the group as a permutation group. Again, see [Paulhus and Rojas 16] for code used. For the Jacobian decompositions, when we write $E^n \times E^m$ we are assuming that $E^n$ corresponds to one factor $B_i^{n_i}$ from (2) and $E^m$ corresponds to a different factor $B_j^{n_j}$ in (2). Our technique does not rule out the possibility that these elliptic curves are in fact isogenous.

**Example 1.** A complete search of genus 12 curves as listed in [Breuer 00] using techniques from Section 2.1 gives no example of a genus 12 curve with a completely decomposable Jacobian. However, we may find one as the quotient of a higher genus curve which has a completely decomposable Jacobian. There is a curve $X$ of genus 29 with the action of $G =$PGL$(2,7) \times C_2$, (where $C_2$ is the cyclic group of order 2) and signature $[0; 2, 4, 6]$. In the Magma or GAP small group databases, this is group $(672, 1254)$.

First we compute the Jacobian decomposition for this curve (2). The Schur index of all characters of this group is 1 and so $n_i$ in (2) will be the dimension of the corresponding irreducible $\mathbb{C}$-representation. For this particular group, all irreducible $\mathbb{C}$-representations have dimensions 1, 6, 7, or 8. To compute the dimensions of the $B_i$ in (2), we must determine $\chi_V$ from (3). The generating vector for this action is computed using using modifications to [Breuer 00] as described in [Paulhus 15]. See Section 3 for more information.

Once we have $\chi_V$, it only remains to compute the irreducible $\mathbb{Q}$-characters using Proposition 2.2, and the inner product in (5). The four linear irreducible $\mathbb{C}$-characters are each irreducible $\mathbb{Q}$-characters, but the inner product of each with $\chi_V$ is 0. There are six irreducible $\mathbb{C}$-characters of degree 6. Two are also irreducible $\mathbb{Q}$-characters, while the other four form two irreducible $\mathbb{Q}$-characters in pairs (using Proposition 2.2 they form two pairs of Galois conjugates). The inner product in (5) is 0 for all but one of these characters– one of the irreducible $\mathbb{C}$-characters which is also an irreducible $\mathbb{Q}$-character. In both degree 7 and 8, the group $G$ admits four irreducible $\mathbb{C}$-characters. All of these are also irreducible $\mathbb{Q}$-characters and when we compute the inner product as in (5) we get 0 for all but one degree 7 character and all but two degree 8 characters.

In all cases where the inner product is greater than 0, it evaluates to 2, hence by (5) the dimension of the $B_i$ are all 1. Plugging all the computed values into (2) produces a Jacobian decomposition of $X$ as

$$JX \sim E^6 \times E^7 \times E^8 \times E^8.$$

The group $G$ has four non-normal subgroups $H$ of order 2. We determine (6) for each subgroup, computing the dimension of the $V_i^H$ by (7). One subgroup $H$ is such that the dimensions of the fixed spaces for the corresponding representations from the decomposition above are all 3. Therefore the Jacobian of the intermediate curve $X_H$ (a genus 12 curve) decomposes as the same four elliptic curves as in the decomposition of $JX$, each one to the power of 3. That is,

$$J(X_H) \sim E^3 \times E^3 \times E^3 \times E^3.$$

Note that Ekedahl and Serre also find a genus 12 example as a quotient of the modular curve $X_0(198)$ of genus 29 by an involution. However, the group in our example is too large to be the automorphism group of this modular curve.

**Example 2.** Using this technique on one of our new examples from Section 2.1, we can generate another example. Consider $G = (720, 767)$ acting on a curve $X$ of genus 61 with signature $[0; 2, 6, 6]$. It has a subgroup $H$ of order 2 such that $X_H$ has genus 30 and a completely decomposable Jacobian. Note that Ekedahl and Serre did not construct an example in this genus.

**Example 3.** Finally consider an example of a higher genus curve which is not completely decomposable, but an intermediate cover produces a lower genus curve which is completely decomposable. There is a genus 101 curve with automorphism group $G = (800, 980)$ and signature $[0; 2, 8, 8]$ whose Jacobian decomposes via (2) as

$$JX \sim E \times A_2 \times E^2 \times \underbrace{E^8 \times \cdots \times E^8}_{12}$$

where $A_2$ is an abelian variety of dimension 2. This group has three subgroups $H$ of order 2 which produce quotients of genus 51. One of those three subgroups produces a decomposition as in (2) where the subspace of the factor above of dimension 2 fixed by $H$ has dimension 0 (as computed via (7)), and thus we get the following complete decomposition:

$$JX_H \sim E \times E^2 \times \underbrace{E^4 \times \cdots \times E^4}_{12}.$$

Again, Ekedahl and Serre did not construct an example in this genus.

## 3. Results

In this section we apply the techniques from Sections 2.1 and 2.2 to find completely decomposable Jacobian varieties (including all the genera previously found in either [Ekedahl and Serre 93] or [Yamauchi 07] except for $g = 113$, 205, 649, and 1297).

Our primary task is to find examples where the dimensions in (2) or (6) above are all 0 or 1. To construct our examples, we must know the automorphism group and signature of curves in high genus. We use three data sources for this information. In [Breuer 00] there are complete lists of automorphism groups and signatures for curves of a given genus up to genus 48. We use his data up through genus 20. For genus 21–101, we use data computed by [Conder 10], giving all automorphism groups of size greater than $4(g-1)$ for a given genus $g$ (this size condition guarantees, in particular, that $g_0$ is 0).

Finally, for genus greater than 101, we use the ideas described in [Conder 14] to find possible automorphism groups corresponding to a few targeted signatures (particularly those signatures which gave us lower genus examples as in Theorem 3.1).

A group $G$ acting on a curve with signature $[0; s_1, \ldots, s_r]$ is equivalent to the existence of a surjective homomorphism $K \twoheadrightarrow G$ where $K$ is a Fuchsian group [Harvey 71] defined as

$$K = \langle x_1, \ldots, x_r | x_1^{s_1} = \cdots = x_r^{s_r} = x_1 \cdots x_r = 1 \rangle.$$

To find examples of large groups acting on curves of $g > 100$, we use the Magma command `LowIndexNormalSubgroup(K,n)` to find all possible normal subgroups of the group $K$ up to index $n$. The quotient of $K$ by these normal subgroups will be the automorphism group of some curve. The genus only depends on the signature and the choice of $n$ (see [Farkas and Kra 92, page 260] ). See [Conder 14] for more details. We will see that these large genus curves give us many new examples.

Notice that the computation of $\chi_V$ in (3) requires knowledge of a generating vector of the action. Modifications to [Breuer 00] give us a way to compute generating vectors if the automorphism group and signature are already known. See [Paulhus 15] for details.

For each of these three data sets and a fixed group $G$ and signature, we first compute the Jacobian decomposition as in (2) and, if this is completely decomposable, we record it. Next we compute all subgroups of $G$ and if any of those produce a quotient of genus still without a known example, we apply the technique of Section 2.2 to determine if this subgroup produces a completely decomposable intermediate cover. Note that from (6), if we take a completely decomposable Jacobian of higher genus, the Jacobian variety corresponding to any intermediate quotient by any subgroup will automatically be completely decomposable.

In our computations, as we increased the genus, we removed from consideration all lower genera we had already found an example for. So our examples for Section 2.2 are just a sample of such curves for a given genus and may not represent all curves of that genus which have decomposable Jacobians realizable through group actions. We chose as our goal demonstrating the usefulness of our technique, and not performing an exhaustive search of all decomposable Jacobians for any known genus.

We divide the results into three sections: those found through the technique in §2.1, those found through the technique in §2.2, and those which give a family of dimension greater than 0 of completely decomposable Jacobians of a given genus.

3.1. **Group algebra decomposition examples.** The new genera, those not included in Ekedahl and Serre's paper, found using the technique in Section 2.1 are given here.

**Theorem 3.1.** *Let $g \in \{36, 46, 81, 85, 91, 193, 244\}$. There is a completely decomposable Jacobian variety of dimension $g$. Moreover, each one corresponds to the Jacobian variety of a curve of genus $g$ with the action of a group $G$ as listed in Table 1. The signature for the action and the decomposition are also listed in the table.*

*Proof.* The proof consists of following the program outlined in Section 2.1. For this we need to find appropriate group actions for the missing genera.

In genus 36 there are, up to topological equivalence, two curves with automorphism group $\mathrm{PGL}(2,7)$ and signature $[0; 2, 6, 8]$. It is possible to classify actions topologically by using the action of the braid group on a generating vector for the action. We do not describe these details here, but references are [Broughton 90], [Harvey 71] and [Völklein 96]. A review of the principal results on this matter and a program in Sage [Stein et al. 15] which computes the non-equivalent actions, can be found in [Muñoz 14], [Behn et al. 15].

To decompose these Jacobian varieties, we need to determine the dimension of the $B_i$ and the values of the $n_i$ in (2). The irreducible $\mathbb{C}$-characters of this group are all irreducible $\mathbb{Q}$-characters, except for two of degree 6, and the Schur index of all characters is one. This means the $n_i$ are just the dimensions of the corresponding irreducible $\mathbb{C}$-representations. To compute the dimension of the $B_i$ we must first compute $\chi_V$ as in (3), using modifications to [Breuer 00] as described in [Paulhus 15] to determine the generating vector. Next, we compute the inner products as in (5). The components which give a non-trivial value for this inner product come from the irreducible $\mathbb{C}$-character of degree 6 which is also an irreducible $\mathbb{Q}$-character, and the two irreducible $\mathbb{Q}$-characters in each of degrees 7 and 8.

The decomposition then follows from (2) and (5):

$$JX \sim E^6 \times E^7 \times E^7 \times E^8 \times E^8.$$

In genus 46 there is one curve, up to topological equivalence, with automorphism group $(324, 69)$ and signature $[0; 2, 6, 18]$. Again, to determine the decomposition, we need to determine the $n_i$ and dimension of the $B_i$ in (2). Once more, the Schur index is 1 for all characters in this group. We compute $\chi_V$ from (3) and then compute the inner product in (5) with each irreducible $\mathbb{Q}$-character.

In this case, the non-zero inner products (which correspond to non-trivial factors in the Jacobian decomposition) come from two separate sets of two linear irreducible $\mathbb{C}$-characters whose sums are irreducible $\mathbb{Q}$-characters (each pair is a pair of Galois conjugates), one set of two degree 2 irreducible $\mathbb{C}$-characters whose sum is also an irreducible $\mathbb{Q}$-character, and seven of the eight irreducible $\mathbb{C}$-characters of degree 6 which are all also irreducible $\mathbb{Q}$-characters. Again, see Proposition 2.2 for how we compute irreducible $\mathbb{Q}$-characters from complex character tables.

This curve, then, has a decomposition

$$JX \sim E \times E \times E^2 \times E^6 \times E^6 \times E^6 \times E^6 \times E^6 \times E^6 \times E^6.$$

In genus 81 a curve $X$ with automorphism group $(1152, 157853)$ and signature $[0; 2, 4, 9]$ has Jacobian decomposition

$$JX \sim E^9 \times E^9 \times E^9 \times E^9 \times E^9 \times E^9 \times E^9 \times E^9 \times E^9.$$

For genus 85 there is a curve $X$ with automorphism group of size 2016 given as [Conder 10]

$$\langle x, y, z | x^2, z^{-1}y^{-1}x, y^4, z^6, y^{-1}zyxz^2yxy^{-1}zy^{-1}z^{-2}xz, yz^{-1}yz^{-1}yz^{-1}xy^2z^{-1}yz^{-1}yz^{-2}\rangle$$

and with signature $[0; 2, 4, 6]$ which has Jacobian decomposition

$$E^6 \times E^7 \times E^8 \times E^8 \times E^{12} \times E^{14} \times E^{14} \times E^{16}.$$

In the genus 81 case, the factors in this decomposition come from nine separate degree 9 irreducible $\mathbb{C}$-characters which are all irreducible $\mathbb{Q}$-characters. In the genus 85 case, the factors in the decomposition come from irreducible $\mathbb{C}$-characters one each of degree 6, 7, 12, and 16, and two each of degree 8 and 14. All of these characters are irreducible $\mathbb{Q}$-characters.

For genus 91 there is a one-dimensional family of curves with automorphism group $G = (432, 686)$ and signature $[0; 2, 2, 2, 12]$. All curves in this family are completely decomposable. Using data from [Conder 10] for genus 91, there is no larger automorphism group which has curves with completely decomposable Jacobians. In particular, no curve in this family has a larger automorphism group.

Finally, for genus 193 there is a curve with automorphism group of size 5760 and signature $[0; 2, 3, 10]$ while in genus 244, the size of the group is $11, 664$ and the signature is $[0; 2, 3, 8]$. Both examples were found using the Magma command `LowIndexNormalSubgroup` to determine the automorphism groups. Here is the presentation of the group for genus 193:

$$\langle x, y, z | x^2, y^3, z^{10}, z^{-1}y^{-1}x, xz^2yz^{-1}xzy^{-1}z^{-2}xzy^{-1}z^{-2}, yz^{-1}xz^4yz^{-1}xy^{-1}z^{-1}xy^{-1}z^{-2}xz^4yz^{-1}xy^{-1}z^{-1}xz,$$
$$z^2y^{-1}z^{-4}xzyz^{-1}xz^2yxzyxz^{-1}xzy^{-1}z^{-3}x\rangle,$$

and here is the presentation for the group of genus 244:

$$\langle x, y, z | x^2, y^3, z^8, z^{-1}y^{-1}x, zyxzyxzyxy^{-1}z^{-1}xy^{-1}z^{-1}xy^{-1}z^{-1}x, z^2yxz^2yxz^2yxz^2yxy^{-1}xy^{-1}z^{-1}xzy^{-1}z^{-1}x\rangle.$$

In all cases, the Schur index is 1. □

In Table 1 we record one example of a curve with completely decomposable Jacobian for each genus found with the technique from Section 2.1. For completeness, we include the genera found by Ekedahl and Serre, or Yamauchi. For each genus, we display an example with the largest automorphism group we found. In many, but not all, cases, this is the largest automorphism group possible for that genus. In the table we include the automorphism group as well as the signature. When possible, we denote the groups as ordered pairs where the first term is the order of the group, and the second term is the group identity number from the Magma or GAP databases. If the order of the group exceeds the allowable sizes for these databases, we have labeled the group as a number (sometimes with a subscript). The number represents the order of the group. If the subscript itself is a number, then the group presentation may be found in data of Conder [Conder 10] where the subscript denotes which of the groups of that order (and with the corresponding signature) in his data it is. If the subscript is a letter (or if there is no subscript at all), the presentation of the group may be found at [Paulhus and Rojas 15].

The final column of the table represents the decomposition as a list of numbers which represent the $n_i$ from (2). Again, we note that it is conceivable that distinct elliptic curve factors in (2) may be isogenous. The new examples from this technique are denoted by a *.

All of our examples come from group actions, while some of Ekedahl and Serre's examples (and the newer work of Yamauchi [Yamauchi 07]) use modular curves. We checked that in the genera where examples were obtained with modular curves in [Ekedahl and Serre 93], our corresponding example was not a modular curve. To determine this, we compared the size of the automorphism group of modular curves of the relevant level, which can be determined by using [Kenku and Momose 88, Theorem 0.1] and [Akbas and Singerman 90, Proposition 2], with the size of the automorphism groups of our examples. Only in $g = 73$ did the sizes match, and in that case we explicitly computed the automorphism group of $X_0(576)$ to determine that it is not the same as our example in Table 1. Notice that our genus 26 example is the well known example of the curve $X(11)$.

Table 1: Curves with completely decomposable Jacobians in genus greater than 10, using group algebra decomposition. The examples are those we found with the largest automorphism group for that genus.

| Genus | Automorphism Group | Signature | Jacobian Decomposition |
|---|---|---|---|
| 11 | $(240, 189)$ | $[0; 2, 4, 6]$ | $5, 6$ |
| 13 | $(360, 121)$ | $[0; 2, 3, 10]$ | $5, 8$ |
| 14 | $(1092, 25)$ | $[0; 2, 3, 7]$ | $14$ |
| 15 | $(504, 156)$ | $[0; 2, 3, 9]$ | $7, 8$ |
| 16 | $(120, 34)$ | $[0; 3, 4, 6]$ | $5, 5, 6$ |
| 17 | $(1344, 814)$ | $[0; 2, 3, 7]$ | $3, 14$ |
| 19 | $(720, 766)$ | $[0; 2, 4, 5]$ | $9, 10$ |
| 21 | $(480, 951)$ | $[0; 2, 4, 6]$ | $5,\ 6,\ 10$ |
| 22 | $(504, 160)$ | $[0; 2, 3, 12]$ | $1, 3, 18$ |
| 24 | $(168, 42)$ | $[0; 3, 4, 7]$ | $3, 6, 7, 8$ |
| 25 | $(576, 1997)$ | $[0; 2, 3, 12]$ | $1, 2, 4, 6, 12$ |
| 26 | $(660, 13)$ | $[0; 2, 3, 11]$ | $5, 10, 11$ |
| 28 | $(1296, 2889)$ | $[0; 2, 3, 8]$ | $2, 8, 18$ |
| 29 | $(672, 1254)$ | $[0; 2, 4, 6]$ | $6, 7, 8, 8$ |
| 31 | $(720, 767)$ | $[0; 2, 4, 6]$ | $5, 6, 8, 12$ |
| 33 | $(1536, 408544637)$ | $[0; 2, 3, 8]$ | $2, 3, 12, 16$ |
| 36* | $(336, 208)$ | $[0; 2, 6, 8]$ | $6, 7, 7, 8, 8$ |
| 37 | $(1728, 31096)$ | $[0; 2, 3, 8]$ | $2, 3, 8, 24$ |
| 41 | $(960, 5719)$ | $[0; 2, 4, 6]$ | $5, 6, 8, 10, 12$ |
| 43 | $(672, 1254)$ | $[0; 2, 4, 8]$ | $6, 7, 7, 7, 8, 8$ |
| 46* | $(324, 69)$ | $[0; 2, 6, 18]$ | $1, 1, 2, \underbrace{6, \ldots, 6}_{7}$ |
| 49 | $(1920, 240996)$ | $[0; 2, 4, 5]$ | $4, 10, 15, 20$ |
| 50 | $(588, 37)$ | $[0; 2, 6, 6]$ | $1, 1, 6, 6, 12, 12, 12$ |
| 55 | $(1296, 3490)$ | $[0; 2, 4, 6]$ | $3, 12, 12, 12, 16$ |
| 57 | $(1344, 11289)$ | $[0; 2, 4, 6]$ | $6, 7, 8, 8, 12, 16$ |
| 61 | $(1440, 4605)$ | $[0; 2, 4, 6]$ | $2, 5, 6, 8, 8, 10, 10, 12$ |
| 65 | $3072_1$ | $[0; 2, 3, 8]$ | $2, 3, 12, 24, 24$ |
| 73 | $(1728, 46270)$ | $[0; 2, 4, 6]$ | $2, 3, 4, 4, 4, 8, 8, 12, 12, 16$ |
| 81* | $(1152, 157853)$ | $[0; 2, 4, 9]$ | $\underbrace{9, \ldots, 9}_{9}$ |
| 82 | $3888_2$ | $[0; 2, 3, 8]$ | $2, 8, 8, 16, 24, 24$ |
| 85 | $4032_1$ | $[0; 2, 3, 8]$ | $8, 14, 18, 21, 24$ |
| 91 | $(432, 686)$ | $[0; 2, 2, 2, 12]$ | $1, 2, 2, 2, \underbrace{4, \ldots, 4}_{21}$ |
| 97 | $3840_1$ | $[0; 2, 4, 5]$ | $4, 10, 15, 20, 24, 24$ |
| 109 | $2592_A$ | $[0; 2, 4, 6]$ | $2, 3, 12, 12, 12, 12, 16, 16, 24$ |
| 121 | $2880$ | $[0; 2, 4, 6]$ | $3, 5, 6, 8, 12, 12, 12, 15, 15, 15, 18$ |

Table 1: (continued)

| Genus | Automorphism Group | Signature | Jacobian Decomposition |
|---|---|---|---|
| 129 | 10752 | $[0; 2, 3, 7]$ | $3, 14, 14, 42, 56$ |
| 145 | 6912 | $[0; 2, 3, 8]$ | $2, 3, 8, 12, 24, 24, 24, 48$ |
| 163 | $2592_C$ | $[0; 2, 4, 8]$ | $1, 2, \underbrace{8, \dots, 8}_{8}, \underbrace{16, \dots, 16}_{6}$ |
| 193* | 5760 | $[0; 2, 3, 10]$ | $5, 8, 15, 15, 15, 15, 30, 30, 30, 30$ |
| 244* | 11664 | $[0; 2, 3, 8]$ | $2, 8, 8, 16, 24, 24, 36, 36, 36, 54$ |
| 257 | $12288_A$ | $[0; 2, 3, 8]$ | $2, 3, 12, \underbrace{24, \dots, 24}_{6}, 48, 48$ |
| 325 | 15552 | $[0; 2, 3, 8]$ | $2, 3, 8, 8, 16, \underbrace{24, \dots, 24}_{6}, 48, 48, 48$ |
| 433 | 5184 | $[0; 2, 6, 6]$ | $1, 1, 2, 2, 3, 4, \underbrace{6, \dots, 6}_{8}, \underbrace{12, \dots, 12}_{31}$ |

Many more examples were found than appear in the paper. We provide tables of all examples we found, not just those of the largest automorphism group order, at [Paulhus and Rojas 15]. For genus up to 20, this is a complete list using this technique for all curves with $g_0 = 0$. For genus 21 −101, this is a complete list for all curves with automorphism group larger than $4(g-1)$. For genus beyond 101 we only list the curves found by strategic searching, and there may be other examples for a given genus.

3.2. **Intermediate cover examples.** Using the technique from Section 2.2, we obtain the following new examples. Notice that we found many more new genera with this new technique.

**Theorem 3.2.** *Let $g \in \{30, 32, 34, 35, 39, 42, 44, 48, 51, 52, 54, 58, 62–64, 67, 69, 71, 72, 79, 80, 89, 93, 95, 103, 105–107, 118, 125, 142, 154, 199, 211, 213\}$. There is a completely decomposable Jacobian variety of dimension $g$. Moreover, each one corresponds to the Jacobian variety of a curve obtained as a quotient by $H \leq G$ of a curve of higher genus with the action of a group $G$.*

*Proof.* We give an outline of the proof for one case, the rest follow similarly. Also recall that in Section 2.2 we gave examples of several other cases, with more details.

Consider the group $G = (1152, 5806)$ acting on a curve $X$ of genus 73 with signature $[0; 2, 4, 8]$. Then using techniques as in Theorem 3.1, $JX$ decomposes into 10 factors (each one a power of an elliptic curve),

$$JX \sim E \times E^2 \times E^2 \times E^4 \times E^8 \times E^8 \times E^8 \times E^8 \times E^{16} \times E^{16}.$$

The Schur index for all the irreducible $\mathbb{C}$-characters is 1 and the first three terms in the decomposition come from sums of pairs of irreducible $\mathbb{C}$-characters of the given degrees (again, by Proposition 2.2), while the remaining factors are all from irreducible $\mathbb{C}$-characters which are also irreducible $\mathbb{Q}$-characters.

Using the technique described in Section 2.2, there is a non-normal subgroup $H$ of $G$ of order 2 such that $X_H$ has genus 35 and has a completely decomposable Jacobian. Using the inner product in (7), dimensions of the subspaces of the representations corresponding to the first and third factor in $JX$ fixed by $H$ are all 0, while the rest are half their values in $JX$. The decomposition of the Jacobian variety of the genus 35 curve is as follows, where $E_i$ corresponds to the $i$th term in the decomposition of $JX$ above:

$$J(X_H) \sim E_2 \times E_4^2 \times E_5^4 \times E_6^4 \times E_7^4 \times E_8^4 \times E_9^8 \times E_{10}^8.$$

□

In Table 2 we give one example for each genus where we found an example through intermediate covers (but not through group actions). We use the same convention for labeling groups as in Table 1. Again, complete lists of data we found are at [Paulhus and Rojas 15]. In this table we also include the genus of the intermediate cover, the genus and automorphism group and signature for the larger curve, the subgroup size and number for the corresponding subgroup $H$ (labeled as Magma does, and recall our convention of converting all groups to permutation groups), and the decomposition of the quotient curve. For ease of notation, we group all factors from (2) of the same dimension together, although they may not be in that order, nor correspond to the order of the decomposition of the high genus curve. For instance, if the decomposition is given as $E^2 \times E^4 \times E^2$, we denote this as $2, 2, 4$.

There are some genera (up to 500) on Ekedahl and Serre's list for which the technique in Section 2.1 cannot identify a curve with completely decomposable Jacobian, and which do not appear in Table 1. The set of such genera is {12, 18, 20, 23, 27, 40, 45, 47, 53, 217}. All these examples may be generated using our second technique of intermediate covers from Proposition 2.4. We also collect this data in Table 2. Again, our new examples are denoted with a $*$.

Table 2: Examples of curves with completely decomposable Jacobians in genus greater than 10 using intermediate coverings.

| $g$ | Large $g$ | Automorphism Group | Signature | Subgroups No., Order | Jacobian Decomposition |
|---|---|---|---|---|---|
| 12 | 49 | $(288, 627)$ | $[0; 2, 2, 2, 6]$ | 28, 4 | $\underbrace{1, \ldots, 1}_{6}, 2, 2, 2$ |
| 18 | 73 | $(1152, 5806)$ | $[0; 2, 4, 8]$ | 35, 4 | 1, 1, 2, 2, 2, 2, 4, 4 |
| 20 | 82 | $3888_2$ | $[0; 2, 3, 8]$ | 13, 4 | 2, 2, 4, 6, 6 |
| 23 | 49 | $(256, 3066)$ | $[0; 2, 2, 2, 8]$ | 9, 2 | $1, 1, 1, \underbrace{2, \ldots, 2}_{10}$ |
| 27 | 55 | $(432, 537)$ | $[0; 2, 2, 2, 4]$ | 6, 2 | $1, 2, \underbrace{3, \ldots, 3}_{8}$ |
| 30* | 61 | $(720, 767)$ | $[0; 2, 6, 6]$ | 5, 2 | 2, 2, 3, 3, 4, 5, 5, 6 |
| 32* | 97 | $2304_6$ | $[0; 2, 3, 12]$ | 10, 3 | 2, 2, 4, 4, 4, 8, 8 |
| 34* | 73 | $(432, 682)$ | $[0; 2, 2, 2, 6]$ | 5, 2 | $1, 1, \underbrace{2, \ldots, 2}_{16}$ |
| 35* | 73 | $(1152, 5806)$ | $[0; 2, 4, 8]$ | 10, 2 | 1, 2, 4, 4, 4, 4, 8, 8 |
| 39* | 81 | $(1152, 157853)$ | $[0; 2, 4, 9]$ | 9, 2 | $\underbrace{4, \ldots, 4}_{6}, 5, 5, 5$ |
| 42* | 129 | $3072_F$ | $[0; 2, 3, 12]$ | 11, 3 | $2, \underbrace{4, \ldots, 4}_{6}, 8, 8$ |
| 44* | 91 | $(432, 686)$ | $[0; 2, 2, 2, 12]$ | 7, 2 | $1, 1, \underbrace{2, \cdots, 2}_{21}$ |
| 45 | 91 | $(432, 686)$ | $[0; 2, 2, 2, 12]$ | 8, 2 | $1, 1, 1, \underbrace{2, \cdots, 2}_{21}$ |
| 47 | 97 | $3840_1$ | $[0; 2, 4, 5]$ | 5, 2 | 2, 4, 7, 10, 12, 12 |
| 48* | 145 | $(1728, 13293)$ | $[0; 2, 6, 6]$ | 12, 3 | $1, 1, \underbrace{2, \ldots, 2}_{7}, \underbrace{4, \ldots, 4}_{8}$ |
| 51* | 101 | $2400_1$ | $[0; 3, 3, 4]$ | 3, 2 | 3, 12, 12, 12, 12 |
| 52* | 109 | $2592_A$ | $[0; 2, 4, 6]$ | 7, 2 | 1, 1, 5, 5, 6, 6, 8, 8, 12 |
| 53 | 109 | $(1296, 2945)$ | $[0; 2, 6, 6]$ | 8, 2 | $1, 1, 1, 2, \underbrace{3, \ldots, 3}_{6}, 6, 6, 6, 6, 6$ |
| 54* | 109 | $(1296, 3498)$ | $[0; 2, 4, 12]$ | 6, 2 | $2, 2, 2, \underbrace{4, \ldots, 4}_{8}, 8, 8$ |
| 58* | 244 | 11664 | $[0; 2, 3, 8]$ | 14, 4 | 2, 2, 4, 6, 6, 8, 8, 8, 14 |
| 62* | 257 | $12288_B$ | $[0; 2, 3, 8]$ | 35, 4 | 2, 4, 4, 6, 6, 8, 8, 12,12 |
| 63* | 193 | 5760 | $[0; 2, 3, 10]$ | 9, 3 | 1, 2, 5, 5, 5, 5, 10, 10, 10, 10 |
| 64* | 325 | 3888 | $[0; 2, 6, 6]$ | 28, 4 | $1, \underbrace{3, \ldots, 3}_{21}$ |

Table 2: (continued)

| $g$ | Large $g$ | Automorphism Group | Signature | Subgroups No., Order | Jacobian Decomposition |
|---|---|---|---|---|---|
| 67* | 145 | $(1728, 32233)$ | $[0; 2, 6, 6]$ | 6, 2 | $1, 1, 2, 3, 3, 3, \underbrace{6, \ldots, 6}_{9}$ |
| 69* | 145 | $(1728, 32233)$ | $[0; 2, 6, 6]$ | 9, 2 | $1, 1, 1, 2, 2, 2, 3, 3, \underbrace{6, \ldots, 6}_{9}$ |
| 71* | 145 | $(1728, 13293)$ | $[0; 2, 6, 6]$ | 8, 2 | $1, 1, \underbrace{3, \ldots, 3}_{7}, \underbrace{6, \ldots, 6}_{8}$ |
| 72* | 325 | 15552 | $[0; 2, 3, 8]$ | 22, 4 | $1, 1, 4, \underbrace{5, \ldots, 5}_{6}, 12, 12, 12$ |
| 79* | 163 | $2592_D$ | $[0; 2, 4, 8]$ | 6, 2 | $1, 3, 3, 4, 4, \underbrace{8, \ldots, 8}_{8}$ |
| 80* | 163 | $2592_C$ | $[0; 2, 4, 8]$ | 5, 2 | $\underbrace{4, \ldots, 4}_{8}, \underbrace{8, \ldots, 8}_{6}$ |
| 89* | 193 | 5760 | $[0; 2, 3, 10]$ | 3, 2 | $4, 5, 5, 5, 10, 15, 15, 15, 15$ |
| 93* | 193 | 2304 | $[0; 2, 2, 2, 3]$ | 11, 2 | $1, 1, 1, 1, 2, 3, 3, 3, \underbrace{4, \ldots, 4}_{12}, 6, 8, 8, 8$ |
| 95* | 193 | 2304 | $[0; 2, 2, 2, 3]$ | 12, 2 | $1, 1, 1, 1, 1, 3, 3, 3, 3, \underbrace{4, \ldots, 4}_{12}, 6, 8, 8, 8$ |
| 103* | 433 | 5184 | $[0; 2, 6, 6]$ | 32, 4 | $1, 1, 1, 1, \underbrace{2, \ldots, 2}_{27}, 3, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6$ |
| 105* | 433 | 5184 | $[0; 2, 6, 6]$ | 49, 4 | $\underbrace{1, \ldots, 1}_{8}, \underbrace{2, \ldots, 2}_{18}, 3, \underbrace{4, \ldots, 4}_{13}, 6$ |
| 106* | 325 | 15552 | $[0; 2, 3, 8]$ | 14, 3 | $1, 2, 2, 5, \underbrace{8, \ldots, 8}_{6}, 16, 16, 16$ |
| 107* | 433 | 5184 | $[0; 2, 6, 6]$ | 39, 4 | $\underbrace{1, \ldots, 1}_{10}, \underbrace{2, \ldots, 2}_{18}, 3, \underbrace{4, \ldots, 4}_{13}, 6$ |
| 118* | 244 | 11664 | $[0; 2, 3, 8]$ | 3, 2 | $1, 3, 3, 8, 11, 11, 18, 18, 18, 27$ |
| 125* | 257 | $12288_B$ | $[0; 2, 3, 8]$ | 6, 2 | $1, 4, 8, 8, 12, 12, 16, 16, 24, 24$ |
| 142* | 433 | 5184 | $[0; 2, 6, 6]$ | 17, 3 | $1, 1, \underbrace{2, \ldots, 2}_{8}, \underbrace{4, \ldots, 4}_{31}$ |
| 154* | 325 | 15552 | $[0; 2, 3, 8]$ | 4, 2 | $1, 1, 3, 3, 8, \underbrace{11, \ldots, 11}_{6}, 24, 24, 24$ |
| 161 | 325 | 15552 | $[0; 2, 3, 8]$ | 5, 2 | $1, 4, 4, 8, \underbrace{12, \ldots, 12}_{6}, 24, 24, 24$ |
| 199* | 433 | 5184 | $[0; 2, 6, 6]$ | 5, 2 | $1, 1, 2, 3, 3, 3, \underbrace{6, \ldots, 6}_{31}$ |
| 211* | 433 | 5184 | $[0; 2, 6, 6]$ | 7, 2 | $1, 1, 2, \underbrace{3, \ldots, 3}_{7}, \underbrace{6, \ldots, 6}_{31}$ |
| 213* | 433 | 5184 | $[0; 2, 6, 6]$ | 9, 2 | $1, 1, 1, 2, 2, 2, \underbrace{3, \ldots, 3}_{6}, \underbrace{6, \ldots, 6}_{31}$ |
| 217* | 433 | 5184 | $[0; 2, 6, 6]$ | 10, 2 | $1, 1, 1, 2, \underbrace{3, \ldots, 3}_{6}, 4, 4, \underbrace{6, \ldots, 6}_{31}$ |

3.3. **Examples of families.** Recall from the proof of Theorem 3.1 that the only completely decomposable Jacobian varieties of dimension 91 discovered using the group algebra technique are a one-dimensional family of curves (so using the group algebra technique only, there is no curve with an automorphism group corresponding to a dimension 0 family in genus 91 having a completely decomposable Jacobian). There are several known examples of families of completely decomposable Jacobians in low genus (see [Frediani et al. 15], [Lange and Rojas 12, Section 4], [Paulhus 08]), and, as we mentioned in the introduction, in [Moonen and Oort 11] the authors asked for examples of special subvarieties such that the generic point is completely decomposable. Our techniques provide a way of finding families where one can look for examples to answer their question.

Here we highlight the genera where we find a one-dimensional (or higher) family of completely decomposable Jacobians of that genus. We elaborate on the question in [Moonen and Oort 11] after the theorem.

**Theorem 3.3.** *Let $g \in \{11\text{–}19, 21\text{–}29, 31, 33\text{–}35, 37, 40, 41, 43\text{–}47, 49, 52, 53, 55, 57, 61, 65, 57, 69, 73, 82, 91, 93, 95, 97, 109, 129, 145, 193\}$. Then there is a dimension one (or larger) family of completely decomposable Jacobians of curves of genus g which can be found using the techniques from Sections 2.1 and 2.2.*

*Proof.* Again, we only demonstrate with a couple of examples. The rest follow in the same way via data listed in Table 3 for those genera found through the technique in Section 2.1, and Table 4 for those found through the technique in Section 2.2. In these tables, for each genus we only give an example of the largest automorphism group we found which leads to a completely decomposable Jacobian (and the highest dimensional family, if that is not the same). Again, for genus greater than 20, we only searched groups of order greater than $4(g-1)$, so there may be other examples of higher dimensional families with completely decomposable Jacobians. All other examples we found appear in the data at [Paulhus and Rojas 15]. For completeness, we have added all examples for genus 3 through 10 curves in the Appendix, only including those corresponding to the action of the full automorphism group [Ries 93]. This data includes many previously known examples.

There is a family of curves of genus 73 with the action of the group $(432, 682)$ with signature $[0; 2, 2, 2, 6]$, see the data at [Paulhus and Rojas 15]. Since this curve is completely decomposable, all quotients by corresponding subgroups $H$ will also be completely decomposable. In particular, this group has a subgroup of order 2 which gives a new example for genus 34.

There are several different group actions on curves of genus 49 giving one-dimensional families of completely decomposable Jacobians. For instance, the group $(256, 3066)$ acts with signature $[0; 2, 2, 2, 8]$ and has a subgroup of order 2 which forms a quotient of genus 23, and the group $(288, 627)$ acting with signature $[0; 2, 2, 2, 6]$ has a subgroup of order 4 which forms a quotient of genus 12. □

Using notation from earlier in the paper, let $G$ be a finite group acting on a curve of genus $g$ with signature $m = [0; s_1, \dots, s_r]$, and generating vector $\theta = (c_1, \dots, c_r)$. For a fixed pair $(m, \theta)$, by moving the branch points of the covering in $\mathbb{P}^1$ one obtains an $(r-3)$-dimensional family of such coverings, and a corresponding family of Jacobians $\mathcal{J}(G, m, \theta)$ of the same dimension. For references see [Frediani et al. 15] or [Völklein 96].

Let $\mathbb{H}_g$ be the Siegel upper half space of complex $g \times g$ symmetric matrices with positive definite imaginary part. The real symplectic group $Sp(2g, \mathbb{R})$ acts transitively on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} * Z = (A + ZC)^{-1}(B + ZD).$$

This action, when considering elements in $Sp(2g, \mathbb{Z})$, identifies Riemann matrices corresponding to isomorphic principally polarized abelian varieties ([Rodríguez 14], [Birkenhake and Lange 04]). Hence $\mathcal{A}_g = Sp(2g, \mathbb{Z}) \backslash \mathbb{H}_g$ is a complex analytic space which parametrizes isomorphism classes of principally polarized abelian varieties of dimension $g$. From the analytic point of view, it corresponds to the moduli space of principally polarized abelian varieties over $\mathbb{C}$ of dimension $g$.

If a subvariety of $\mathcal{A}_g$ is the image of one orbit of an algebraic subgroup of $Sp(2g, \mathbb{R})$ under this action, then we say the subvariety is a *special subvariety*. Special subvarieties have some interesting geometric properties. For instance, *special points*, i.e., special subvarieties of dimension zero, correspond to varieties of

*CM-type*, which are varieties with interesting endomorphism rings. For details, we refer the reader to [Moonen and Oort 11] and [Frediani et al. 15].

Denote by $Z(G, m, \theta)$ the closure of the family $\mathcal{J}(G, m, \theta)$ in $\mathcal{A}_g$. It is a $r-3$ dimensional subvariety of $\mathcal{A}_g$. The goal is to determine if it is a special subvariety of $\mathcal{A}_g$. In [Frediani et al. 15, Thms. 1.4, 3.9, Lemma 3.8] there is a nice characterization of when $Z(G, m, \theta)$ is a special subvariety. Their criterion is as follows, if $JX \in \mathcal{J}(G, m, \theta)$ is one of the Jacobians in the family corresponding to one covering $X \to X/G \cong \mathbb{P}^1$, consider the symplectic representation $\rho : G \to Sp(2g, \mathbb{Z})$ of $G$ induced by the action of $G$ in the lattice of $JX$, or equivalently induced by the action of $G$ in the first homology group $H_1(X, \mathbb{Z})$ (see [Behn et al. 13] for details). Let $\mathbb{H}_g^G$ be the set of fixed points of $G$ in $\mathbb{H}_g$, and denote by $N$ the dimension of the irreducible component containing $\mathcal{J}(G, m, \theta)$ in $\mathbb{H}_g^G$. Both the isomorphism class of $\rho$ and the dimension $N$ depend only on the fixed pair $(m, \theta)$ for $G$, not on the particular element $JX$, nor the particular covering $X \to \mathbb{P}^1$, of the family. If the dimension $N$ equals the dimension of $\mathcal{J}(G, m, \theta)$, which is $r-3$, then $Z(G, m, \theta)$ is a special subvariety of $\mathcal{A}_g$ that is contained in the closure of the Torelli locus $\mathcal{T}_g$, and which intersects the (open) Torelli (or Jacobian) locus $\mathcal{T}_g^0$ non-trivially.

Given a pair $(m, \theta)$ for a fixed $G$, using [Behn et al. 13] one can find the dimension of $\mathbb{H}_g^G$, although it is computationally expensive and it is not easy to find the dimension of the specific irreducible component containing the family considered, unless $\mathbb{H}_g^G$ is irreducible (which can be determined using Magma). Nevertheless, useful code is provided in [Frediani et al. 15] which can compute the dimension $N$ for low genus examples.

Our Table 3 contains examples of families found using group actions, so we can apply the criterion of [Frediani et al. 15] to determine if they correspond to special subvarieties. We remark that these families could correspond to special subvarieties even if they do not satisfy the criterion. Moreover, in Table 4, we give examples of families of completely decomposable Jacobian varieties arising from intermediate coverings, in which case the criterion of [Frediani et al. 15] cannot be directly applied, since one has here $X \to X/H \to X/G \cong \mathbb{P}^1$ where the last covering of $\mathbb{P}^1$ is not (in general) Galois. It is a work in progress to adjust the criterion to this situation.

We show with one example how the families on Table 3 correspond to special subvarieties. Let $G$ be the alternating group $A_4$, acting on a curve of genus 4 with signature $m = [0; 2, 3, 3, 3]$. We have then a one-dimensional family $\mathcal{J}$ of Jacobians. Using [Behn et al. 13] we determine that the dimension of $\mathbb{H}_4^G$ is also 1. Therefore, according to [Frediani et al. 15], the closure $Z$ of $\mathcal{J}$ is a special subvariety of $\mathcal{A}_4$ contained in $\mathcal{T}_4$ and such that $Z \cap \mathcal{T}_4^0 \neq \emptyset$.

Using the group algebra decomposition (2), we conclude that the elements in $\mathcal{J}$ (hence in $\mathbb{H}_4^G$) decompose as $E \times E_1^3$ (see the Appendix). Therefore this illustrates a special case of [Moonen and Oort 11, Question 6.6]. Notice that $E$ corresponds to an irreducible representation $\varphi$ of $G$ such that $G/\ker(\varphi) \cong \mathbb{Z}/3\mathbb{Z}$, hence $E$ has the action of the cyclic group of order 3 and thus it is fixed along the family. This family is one of the special subvarieties found in [Frediani et al. 15, Table 2].

Table 3: Examples of families of completely decomposable curves found through the group algebra method.

| $g$ | Automorphism Group | Signature | Jacobian Decomposition |
|---|---|---|---|
| 11 | (24, 14) | [0; 2, 2, 2, 2, 6] | 1, 1, 1, 2, 2, 2, 2 |

Table 3: (continued)

| $g$ | Automorphism Group | Signature | Jacobian Decomposition |
|---|---|---|---|
| | (48, 38) | [0; 2, 2, 2, 12] | 1, 2, 2, 2, 4 |
| 13 | (144, 183) | [0; 2, 2, 2, 3] | 2, 2, 3, 6 |
| | (48, 51) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 2, 2, 2, 2 |
| 15 | (48, 48) | [0; 2, 2, 4, 6] | 1, 2, 3, 3, 3, 3 |
| 16 | (36, 13) | [0; 2, 2, 2, 2, 6] | $1, 1, \underbrace{2, \dots, 2}_{7}$ |
| 17 | (192, 956) | [0; 2, 2, 2, 3] | 2, 3, 6, 6 |
| | (64, 211) | [0; 2, 2, 2, 2, 2] | $1, 1, 1, 1, 1, \underbrace{2, \dots, 2}_{6}$ |
| 19 | (144, 109) | [0; 2, 2, 2, 4] | 1, 3, 3, 6, 6 |
| | (72, 49) | [0; 2, 2, 2, 2, 2] | $1, 1, 1, \underbrace{2, \dots, 2}_{8}$ |
| 25 | (288,847) | [0; 2, 2, 2, 3] | 2, 2, 3, 4, 6, 8 |
| 28 | (324,124) | [0; 2, 2, 2, 3] | 2, 2, 2, 4, 6, 6, 6 |
| 31 | (144, 154) | [0; 2, 2, 2, 12] | $1, 2, 2, 2, \underbrace{4, \dots, 4}_{6}$ |
| 33 | (384,18136) | [0; 2, 2, 2, 3] | 3, 3, 3, 8, 8, 8 |
| 37 | (432,748) | [0; 2, 2, 2, 3] | 2, 2, 2, 3, 4, 6, 6, 12 |
| 49 | (576, 8653) | [0; 2, 2, 2, 3] | 2, 2, 3, 3, 6, 6, 9, 9, 9 |
| 55 | (432, 537) | [0; 2, 2, 2, 4] | $1, 3, 3, \underbrace{6, \dots, 6}_{8}$ |
| 61 | (288, 629) | [0; 2, 2, 2, 12] | $1, \underbrace{2, \dots, 2}_{6}, \underbrace{4, \dots, 4}_{12}$ |
| 65 | (768, 1090018) | [0; 2, 2, 2, 3] | $2, 3, 3, 3, 6, \underbrace{8, \dots, 8}_{6}$ |
| 73 | (576, 4322) | [0; 2, 2, 2, 4] | $1, 2, 2, \underbrace{4, \dots, 4}_{9}, 8, 8, 8, 8$ |
| 82 | (972, 474) | [0; 2, 2, 2, 3] | $2, 2, 2, 4, \underbrace{6, \dots, 6}_{6}, 12, 12, 12$ |
| 91 | (432, 686) | [0; 2, 2, 2, 12] | $1, 2, 2, 2, \underbrace{4, \dots, 4}_{21}$ |
| 97 | (1152, 157665) | [0; 2, 2, 2, 3] | 2, 2, 3, 3, 3, 6, 6, 6, 6, 8, 8, 8, 8, 12, 16 |
| 109 | (1296, 2940) | [0; 2, 2, 2, 3] | $2, 2, 2, 3, 4, \underbrace{6, \dots, 6}_{8}, 12, 12, 12, 12$ |
| 129 | 1536 | [0; 2, 2, 2, 3] | $2, 3, 3, 3, 6, 6, 6, 6, 6, 8, 8, \underbrace{12, \dots, 12}_{6}$ |
| 145 | (1728, 46119) | [0; 2, 2, 2, 3] | $2, 2, 3, 3, 3, \underbrace{6, \dots, 6}_{8}, \underbrace{12, \dots, 12}_{7}$ |
| 193 | 2304 | [0; 2, 2, 2, 3] | $2, 2, 3, 3, 3, 6, 6, 6, 6, 6, \underbrace{8, \dots, 8}_{12}, 12, 16, 16, 16$ |

Table 4: Families of completely decomposable curves found through the intermediate cover method.

| $g$ | Large $g$ | Automorphism Group | Signature | Subgroup No., Order | Jacobian Decomposition |
|---|---|---|---|---|---|
| 12 | 49 | (288, 627) | [0; 2, 2, 2, 6] | 29, 4 | 1, 1, 1, 1, 2, 2, 2, 2 |
| 14 | 145 | (1728, 46119) | [0; 2, 2, 2, 3] | 168, 8 | $\underbrace{1, \dots, 1}_{11}, 3$ |
| 18 | 145 | (1728, 46119) | [0; 2, 2, 2, 3] | 152, 8 | $\underbrace{1, \dots, 1}_{11}, 2, 2, 3$ |
| 21 | 129 | 1536 | [0; 2, 2, 2, 3] | 128, 6 | $\underbrace{1, \dots, 1}_{7}, \underbrace{2, \dots, 2}_{7}$ |
| 22 | 193 | 2304 | [0; 2, 2, 2, 3] | 132, 8 | $\underbrace{1, \dots, 1}_{7}, \underbrace{2, \dots, 2}_{6}, 3$ |
| 23 | 49 | (256, 3066) | [0; 2, 2, 2, 8] | 9, 2 | $1, 1, 1, \underbrace{2, \dots, 2}_{10}$ |
| 24 | 49 | (288, 627) | [0; 2, 2, 2, 6] | 8, 2 | $\underbrace{1, \dots, 1}_{8}, \underbrace{2, \dots, 2}_{8}$ |
| 26 | 109 | (1296, 2940) | [0; 2, 2, 2, 3] | 22, 4 | $\underbrace{1, \dots, 1}_{7}, 2, 2, 2, 2, 2, 2, 3, 3, 3$ |
| 27 | 55 | (432, 537) | [0; 2, 2, 2, 4] | 6, 2 | $1, 2, \underbrace{3, \dots, 3}_{8}$ |
| 29 | 97 | (1152, 157665) | [0; 2, 2, 2, 3] | 13, 3 | $1, 1, 1, \underbrace{2, \dots, 2}_{8}, 4, 6$ |

Table 4: (continued)

| $g$ | Large $g$ | Automorphism Group | Signature | Subgroup No., Order | Jacobian Decomposition |
|---|---|---|---|---|---|
| 33 | 193 | 2304 | [0; 2, 2, 2, 3] | 74, 6 | $\underbrace{1,\ldots,1}_{7},\underbrace{2,\ldots,2}_{13}$ |
| 34 | 73 | (432, 682) | [0; 2, 2, 2, 6] | 5, 2 | $1,1,\underbrace{2,\ldots,2}_{16}$ |
| 35 | 145 | (1728, 46119) | [0; 2, 2, 2, 3] | 54, 4 | $\underbrace{1,\ldots,1}_{10},2,2,2,3,4,4,4,4$ |
| 40 | 82 | (972, 474) | [0; 2, 2, 2, 3] | 4, 2 | $1,1,2,\underbrace{3,\ldots,3}_{6},6,6,6$ |
| 41 | 129 | 1536 | [0; 2, 2, 2, 3] | 15, 3 | $1,1,1,\underbrace{2,\ldots,2}_{7},\underbrace{4,\ldots,4}_{6}$ |
| 43 | 193 | 2304 | [0; 2, 2, 2, 3] | 34, 4 | 1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 4, 6, 8 |
| 44 | 91 | (432, 686) | [0; 2, 2, 2, 12] | 7, 2 | $1,1,\underbrace{2,\cdots,2}_{21}$ |
| 45 | 91 | (432, 686) | [0; 2, 2, 2, 12] | 8, 2 | $1,1,1,\underbrace{2,\cdots,2}_{21}$ |
| 46 | 109 | (1296, 2940) | [0; 2, 2, 2, 3] | 3, 2 | $1,1,1,\underbrace{2,\cdots,2}_{8},3,6,6,6,6$ |
| 47 | 97 | (1152, 157665) | [0; 2, 2, 2, 3] | 10, 2 | 1, 1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 4, 4, 6, 8 |
| 52 | 109 | (1296, 2940) | [0; 2, 2, 2, 3] | 5, 2 | $1,1,1,2,2,\underbrace{3,\ldots,3}_{7},6,6,6,6$ |
| 53 | 109 | (1296, 2940) | [0; 2, 2, 2, 3] | 6, 2 | $1,1,1,2,\underbrace{3,\ldots,3}_{8},6,6,6,6$ |
| 57 | 193 | 2304 | [0; 2, 2, 2, 3] | 15, 3 | $1,1,1,\underbrace{2,\ldots,2}_{16}\,4,6,6,6$ |
| 67 | 145 | (1728, 46119) | [0; 2, 2, 2, 3] | 11, 2 | $1,1,1,1,1,2,2,2,2,3,3,3,3,\underbrace{6,\ldots,6}_{7}$ |
| 69 | 145 | (1728, 46119) | [0; 2, 2, 2, 3] | 12, 2 | $1,1,1,1,2,\underbrace{3,\ldots,3}_{7},\underbrace{6,\ldots,6}_{7}$ |
| 93 | 193 | 2304 | [0; 2, 2, 2, 3] | 11, 2 | $1,1,1,1,2,3,3,3,\underbrace{4,\ldots,4}_{12},6,8,8,8$ |
| 95 | 193 | 2304 | [0; 2, 2, 2, 3] | 12, 2 | $1,1,1,1,1,3,3,3,3,\underbrace{4,\ldots,4}_{12},6,8,8,8$ |

## 4. Complications

The techniques described above do not necessarily yield the finest decomposition. In (2), it is possible that the $B_i$ may decompose further. Thus there may be examples using a finer decomposition which fill other gaps in Ekedahl and Serre's list. Moreover, it is also possible that $B_i \sim B_j$ even if $i \neq j$ in (2). In this case, we are exhibiting more factors than is necessary up to isogeny.

Computationally, finding automorphism groups and signatures in high genus is resource heavy. The memory requirements for the Magma command `LowIndexNormalSubgroups` limit our ability to use this command to find other examples in higher genus, or to fill remaining gaps using the intermediate cover technique. For example, because of computational constraints we could not find examples in genus 649 and 1297 as Ekedahl and Serre do. We are optimistic that, given sufficient computational resources, the techniques we describe above could produce numerous additional new examples.

## 5. Appendix

Here we provide all examples of families of curves for genus 3–10 which have completely decomposable Jacobians. These were found by searching all Breuer's data for these genera, and then removing those groups that were not the full automorphism group for the given family [Ries 93]. Some of the examples in this table were known before [Frediani et al. 15].

Table 5: Family of completely decomposable curves found through group algebra method for genus 3-10.

| $g$ | Automorphism Group | Signature | Jacobian Decomposition |
|---|---|---|---|
| 3 | (4, 2) | [0; 2, 2, 2, 2, 2, 2] | 1, 1, 1 |
| | (6, 1) | [0; 2, 2, 2, 2, 3] | 1, 2 |
| | (8, 2) | [0; 2, 2, 4, 4] | 1, 1, 1 |
| | (8, 5) | [0; 2, 2, 2, 2, 2] | 1, 1, 1 |
| | (12, 4) | [0; 2, 2, 2, 6] | 1, 2 |
| | (16, 11) | [0; 2, 2, 2, 4] | 1, 2 |
| | (16, 13) | [0; 2, 2, 2, 4] | 1, 2 |
| | (18, 3) | [0; 2, 2, 2, 2, 2] | 1, 2 |
| | (24, 12) | [0; 2, 2, 2, 3] | 3 |
| 4 | (8, 3) | [0; 2, 2, 2, 2, 4] | 1, 1, 2 |
| | (12, 3) | [0, 2, 3, 3, 3] | 1, 3 |
| | (12, 4) | [0; 2, 2, 3, 6] | 2, 2 |
| | (12, 4) | [0; 2, 2, 2, 2, 2] | 1, 1, 2 |
| | (24, 12) | [0; 2, 2, 2, 4] | 1, 3 |
| | (36, 10) | [0; 2, 2, 2, 3] | 2, 2 |
| 5 | (8, 5) | [0; 2, 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1 |
| | (12, 4) | [0; 2, 2, 2, 2, 3] | 1, 2, 2 |
| | (16, 3) | [0; 2, 2, 4, 4] | 1, 2, 2 |
| | (16, 11) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 2 |
| | (16, 11) | [0; 2, 2, 2, 2, 2] | 1, 2, 2 |
| | (16, 14) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1 |
| | (24, 8) | [0; 2, 2, 2, 6] | 1, 2, 2 |
| | (24, 14) | [0; 2, 2, 2, 6] | 1, 2, 2 |
| | (32, 27) | [0; 2, 2, 2, 4] | 1, 2, 2 |
| | (32, 28) | [0; 2, 2, 2, 4] | 1, 2, 2 |
| | (32, 43) | [0; 2, 2, 2, 4] | 1, 4 |
| | (48, 48) | [0; 2, 2, 2, 3] | 2, 3 |
| 6 | (12, 4) | [0; 2, 2, 2, 2, 6] | 1, 1, 2, 2 |
| | (24, 12) | [0; 2, 2, 3, 4] | 3, 3 |
| 7 | (8, 5) | [0; 2, 2, 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 1, 1 |
| | (16, 11) | [0; 2, 2, 2, 2, 4] | 1, 1, 1, 2, 2 |
| | (18, 4) | [0; 2, 2, 2, 2, 3] | 1, 2, 2, 2 |
| | (24, 13) | [0; 2, 2, 3, 6] | 1, 3, 3 |
| | (24, 14) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 2, 2 |
| | (32, 43) | [0; 2, 2, 2, 8] | 1, 2, 4 |
| | (36, 10) | [0; 2, 2, 2, 6] | 1, 2, 4 |
| | (48, 38) | [0; 2, 2, 2, 4] | 1, 2, 4 |
| | (48, 48) | [0; 2, 2, 2, 4] | 1, 3, 3 |
| 8 | (24, 12) | [0; 2, 3, 3, 4] | 2, 3, 3 |
| 9 | (16, 11) | [0; 2, 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 2, 2 |
| | (16, 14) | [0; 2, 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 1, 1, 1, 1 |
| | (24, 14) | [0; 2, 2, 2, 2, 3] | 1, 2, 2, 2, 2 |
| | (32, 6) | [0; 2, 2, 4, 4] | 1, 2, 2, 4 |
| | (32, 27) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 2, 2, 2 |
| | (32, 34) | [0; 2, 2, 2, 2, 2] | 1, 2, 2, 2, 2 |
| | (32, 43) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 2, 4 |
| | (32, 46) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 2, 2 |
| | (32, 49) | [0; 2, 2, 2, 2, 2] | 1, 1, 1, 1, 1, 4 |
| | (48, 38) | [0; 2, 2, 2, 6] | 1, 2, 2, 4 |
| | (48, 43) | [0; 2, 2, 2, 6] | 1, 2, 2, 2, 2 |
| | (48, 48) | [0; 2, 2, 2, 6] | 3, 3, 3 |
| | (64, 73) | [0; 2, 2, 2, 4] | 1, 2, 2, 2, 2 |
| | (64, 128) | [0; 2, 2, 2, 4] | 1, 2, 2, 4 |
| | (64, 134) | [0; 2, 2, 2, 4] | 1, 2, 2, 4 |
| | (64, 135) | [0; 2, 2, 2, 4] | 1, 2, 2, 4 |
| | (64, 138) | [0; 2, 2, 2, 4] | 1, 2, 2, 4 |
| | (64, 140) | [0; 2, 2, 2, 4] | 1, 2, 2, 4 |
| | (64, 177) | [0; 2, 2, 2, 4] | 1, 4, 4 |
| | (96, 193) | [0; 2, 2, 2, 3] | 2, 3, 4 |
| | (96, 227) | [0; 2, 2, 2, 3] | 3, 3, 3 |
| 10 | (36, 10) | [0; 2, 2, 3, 6] | 2, 2, 2, 4 |
| | (36, 13) | [0; 2, 2, 3, 6] | 2, 2, 2, 2, 2 |
| | (36, 10) | [0; 2, 2, 2, 2, 2] | 1, 1, 2, 2, 4 |
| | (36, 13) | [0; 2, 2, 2, 2, 2] | 1, 1, 2, 2, 2, 2 |
| | (48, 29) | [0; 2, 2, 2, 8] | 1, 2, 3, 4 |
| | (72, 15) | [0; 2, 2, 2, 4] | 1, 3, 6 |
| | (72, 40) | [0; 2, 2, 2, 4] | 2, 4, 4 |
| | (72, 43) | [0; 2, 2, 2, 4] | 1, 3, 6 |
| | (108, 17) | [0; 2, 2, 2, 3] | 2, 2, 6 |
| | (108, 40) | [0; 2, 2, 2, 3] | 2, 2, 2, 4 |

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