MAT499
Prof. Marc Chamberland

LU Matrix Decompositions

David Koychev, Hanh Pham, and Kaiqian Zhang

GRINNELL COLLEGE

Introduction

LU decomposition factors a matrix into the product of a lower-triangular matrix (L) and an upper-triangular matrix (U). The goal of this research is to use LU decomposition to build new identities for some special matrices. We work on matrices that arose in the study of three-term recurrence relations, orthogonal polynomials, graph theory, Euler numbers, Bernoulli numbers, and discrete Fourier transformation. We also use Maple (a symbolic computational software) and OEIS (Online Encyclopedia of Integer Sequence) to assist our computations and sequence identification.

Our approach follows the methodology used in Prof. Chamberland's paper:

- perform LU decomposition on a matrix
- identify the patterns of the resulting L and U matrices and make conjectures
- build new identities for the original matrix
- make possible interpretations
- write up formal proofs if possible

Results

1. General LU Decomposition

Knowing both L and U enables us to reconstruct the original matrix A using the formula:

$$A_{ij} = \sum_{k=1}^{\min(i,j)} L_{ik} U_{kj}$$

In fact, most of the identities obtained in this MAP are examples of this formula.
When all of the principal minors of the initial matrix are nonsingular. We can express L and U as the ratio of determinants of certain minors of the original matrix. Let $A(k)$ is the principal minor of A composed from the first k rows and columns. Let $A_i(k)_c$ is the submatrix of A where the k-th column of A is replaced by the i-th column. Similarly the k-th row of A is replaced by j-th row in $A_j(k)_r$. Then the following is true:

$$L_{ij} = \frac{|A_i(j)_r|}{|A(j)|} \qquad U_{ij} = \frac{|A_j(i)_c|}{|A(i-1)|}$$

In most of the cases this form is not very practical computationally, but it can lead to interesting identities concerning other simplified forms of the same LU decomposition. This representation is built upon Householder's work.

2.Three-term recurrence relations

Suppose we have a three-term recurrence relation $a_k = c_1 a_{(k-1)} + c_2 a_{(k-2)}$ where c_1 and c_2 are constants, with initial conditions a_1 and a_2. We consider an associated Hankel matrix A by letting (i, j) entry be $a_{(i+j-1)}$.

$$A = \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ c_1&1&0&0&0\\ {c_1}^2+c_2 & c_1 & 1&0&0\\ {c_1}^3+2c_1c_2 & {c_1}^2+c_2 & 0&1&0\\ {c_1}^4+3{c_1}^2c_2+{c_2}^2 & c_1({c_1}^2+2c_2) & 0&0&1 \end{bmatrix} \begin{bmatrix} 1 & c_1 & {c_1}^2+c_2 & {c_1}^3+2c_1c_2 & {c_1}^4+3{c_1}^2c_2+{c_2}^2\\ 0 & c_2 & c_1c_2 & {c_1}^2c_2+{c_2}^2 & {c_1}^3c_2+2c_1{c_2}^2\\ 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix}$$

Applying LU decomposition to the matrix A gives a L of only first two columns having nonzero entries and the diagonal with 1s, and a U of only the first two rows having nonzero entries.
We also have the Fibonacci sequence $f: \mathbb{N} \to \mathbb{R}$, in which $f(1) = 0$, $f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$.
With the help of OEIS, we identify the pattern that (i,j) entries of L and U are :

$$L_{i,j} = \begin{cases} \frac{a_i}{a_1} & j=1 \\ f(i) & j=2 \end{cases} \quad ; \quad U_{i,j} = \begin{cases} a_j & i=1 \\ f(j)\cdot\left(\frac{{a_1}^2+a_1a_2-{a_2}^2}{a_1} + (c_2-1)a_1 + (c_1-1)a_2\right) & i=2 \\ 0 & i\ge 3 \end{cases}.$$

It is not hard to prove the pattern by using the characteristic equation of the sequence.

In the LU decomposition we can notice that the nonzero entries of U are on the first 2rows and the offdiagonal nonzero entries of L are only in the first two columns. Referring back to the General Lu decomposition formula this suggested that the determinants of the 3 by 3 or bigger minors are 0. Indeed some further algebra with the Binet's formulae verified this.

Thus if we have the following minor of A

$$\begin{bmatrix} a_i & a_{i+k} & a_{i+l} \\ a_{i+m} & a_{i+k+m} & a_{i+l+m} \\ a_{i+n} & a_{i+k+n} & a_{i+l+n} \end{bmatrix}$$

Its determinant will be 0 which produces the following identity:

$$a_i a_{i+k+m} a_{i+l+n} - a_i a_{i+k+n} a_{i+l+m} - a_{i+k} a_{i+m} a_{i+l+n} + a_{i+k} a_{i+n} a_{i+l+m} + a_{i+l} a_{i+m} a_{i+k+n} - a_{i+l} a_{i+n} a_{i+k+m} = 0 .$$

We can also produce arbitrarily long identities when considering bigger minors. Moreover this relations seem to hold for the associated Hankel matrices with Chebyshev Polynomials.

3.Orthogonal polynomials

3a Legendre Polynomials

Orthogonal polynomials, which also satisfy three-term recurrence relations, will produce lovely formula under LU decomposition. Legendre polynomial is one of well-known orthogonal polynomials involving Rodrigue's formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$. We construct a matrix M based on the Rodrigue's formula by letting the (i, j) entry be :

$$M_{ij} = \begin{cases} (x^2-1)^{j-1} & i=1 \\ \frac{d^{i-1}}{dx^{i-1}}[(x^2-1)^{j-1}] & \text{otherwise} \end{cases}.$$

$$M = \begin{bmatrix} 1 & x^2-1 & (x^2-1)^2 & (x^2-1)^3 & (x^2-1)^4 \\ 0 & 2x & (4x^2-4)x & 6(x^2-1)^2x & 8(x^2-1)^3x \\ 0 & 2 & 12x^2-4 & (24x^2-24)x^2+6(x^2-1)^2 & 48(x^2-1)^2x^2+8(x^2-1)^3 \\ 0&0&24x & 48x^3+(72x^2-72)x & (192x^2-192)x^3+144(x^2-1)^2x \\ 0&0&24&360x^2-72 & 384x^4+(1152x^2-1152)x^2+144(x^2-1)^2 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&x^{-1}&1&0&0\\0&0&3x^{-1}&1&0\\0&0&3x^{-2}&6x^{-1}&1 \end{bmatrix} \begin{bmatrix} 1 & x^2-1 & (x^2-1)^2 & (x^2-1)^3 & (x^2-1)^4 \\ 0 & 2x & (4x^2-4)x & 6(x^2-1)^2x & 8(x^2-1)^3x \\ 0&0&8x^2 & 24x^4-24x^2 & 48x^6-96x^4+48x^2 \\ 0&0&0&48x^3 & (192x^2-192)x^3 \\ 0&0&0&0&384x^4 \end{bmatrix}$$

The LU decomposition yields L and U with the terms:

$$L_{i,j} = \begin{cases} 0 & i<j \\ \frac{(2i-2j)!\binom{i-1}{2i-2j}}{2^{i-j}(i-j)!}x^{j-i} & i\ge j \end{cases} \quad ; \quad U_{i,j} = \begin{cases} 0 & i>j \\ \frac{2^{i-1}(j-1)!}{(j-i)!}(x^2-1)^{j-i}x^{i-1} & i\le j \end{cases}.$$

This leads to an identity $M_{ij} = \sum_{k=1}^{\min(i,j)} L_{ik}U_{kj}$, which is equivalent to

$$\frac{d^{i-1}}{dx^{i-1}}[(x^2-1)^{j-1}] = \sum_{k=1}^{\min(i,j)} (j-1)!\frac{(2i-2k)!}{(i-k)!(j-k)!}\binom{i-1}{2i-2k}2^{2k-i-1}x^{2k-i-1}(x^2-1)^{j-k}.$$

We are aware that when i=j, we get part of Legendre polynomial, which delivers

$$\frac{d^{i-1}}{dx^{i-1}}[(x^2-1)^{i-1}] = \sum_{k=1}^{i}(i-1)!\binom{2i-2k}{i-k}\binom{i-1}{2i-2k}2^{2k-i-1}x^{2k-i-1}(x^2-1)^{i-1}.$$

By substituting all the i-1 with n and multiplying both sides by $\frac{1}{2^n n!}$, the left-hand side boils down to Legendre polynomials. Therefore, we engender a new formula for Legendre polynomials:

$$P_n(x) = \frac{1}{2^n}\sum_{k=\lceil \frac{n}{2} \rceil}^{n}\binom{2n-2k}{n-k}\binom{n}{2n-2k}(2x)^{2k-n}(x^2-1)^{n-k}.$$

We proved this formula by showing the equivalence to $P_n(x) = \frac{1}{2^n}\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^l\binom{n}{l}\binom{2n-2l}{n}x^{(n-2l)}$, a known formula for Legendre polynomials (Koepf, 1998).

3b Laguerre Polynomials

We use similar method in exploring another orthogonal polynomials-- Laguerre polynomials. We derive a matrix N from its Rodrigue's formula by letting the (i, j) entry be

$$N_{ij} = \begin{cases} e^{-x}x^{j-1} & i=1 \\ \frac{d^{i-1}}{dx^{i-1}}[e^{-x}x^{j-1}] & \text{otherwise} \end{cases}.$$

The LU decomposition of this matrix produces L and U with the pattern

$$L_{i,j} = (-1)^{i+j}\binom{i-1}{j-1} \quad ; \quad U_{i,j} = (i-1)!\binom{j-1}{i-1}x^{j-i}e^{-x}.$$

This follows the identity that

$$\frac{d^{i-1}}{dx^{i-1}}(e^{-x}x^{j-1}) = \sum_{k=1}^{\min(i,j)} L_{i,k}U_{k,j} = \sum_{k=1}^{\min(i,j)}(-1)^{i+k}\binom{i-1}{k-1}\binom{j-1}{k-1}(k-1)!x^{j-k}e^{-x}.$$

A particular case of the identity involving Laguerre polynomials is when *i*=*j*, we have

$$\frac{d^{i-1}}{dx^{i-1}}(e^{-x}x^{i-1}) = \sum_{k=1}^{i}(-1)^{i+k}\binom{i-1}{k-1}^2(k-1)!x^{i-k}e^{-x}.$$

And we use some algebraic to prove our conjectured Laguerre polynomials formula.

4. Example with Vandermonde matrix

In Vandermonde matrix the i-th row consists of the consecutive powers of single variable x_i

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

The LU decomposition was originally performed by Oruc Halil and yielded the following terms:

$$L_{ij} = \prod_{l=1}^{j-1}\frac{x_i - x_l}{x_j - x_l}$$

$$U_{ij} = h_{j-i}(x_1, x_2, \dots x_i)\prod_{l=1}^{i-1} x_i - x_l = h_{j-i}(x_1, x_2, \dots x_i)U_{ii}$$

In these expressions $h_k(x_1, x_2 \dots x_n)$ is the complete symmetric polynomial of variables $\{x_1, x_2 .. x_n\}$ sums monomials of these variables of total degree k : $h_k(x_1, x_2 \dots x_n) = \sum_{1\le i_1 \le i_2 \dots \le i_n \le n} x_{i_1}x_{i_2}\dots x_{i_n}$
That Decomposition produces the following formula:

$$V_{ij} = h_{j-1}(x_i) = x_i^{j-1} = \sum_{k=1}^{\min i,j} h_{j-k}(x_1, x_2 \dots x_k)\prod_{l=1}^{k-1}(x_i - x_l)$$

A particular example of Vandermonde matrix used widely in the discrete Fourier transformation is where the variables $\{x_1, x_2 .. x_n\}$ are powers on the n-th root of unity q.

$$F = \begin{bmatrix} 1 & q & q^2 & \dots & q^{n-1} \\ 1 & q^2 & q^4 & \dots & q^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & q^{(n-1)} & q^{2(n-1)} & \dots & q^{(n-1)(n-1)} \end{bmatrix}$$

Although the substitution produces directly a nice identity for L, the formula for U involved identifying the harmonic polynomials as q-binomials using combinatorial and some subsequent algebraic manipulations:

$$L_{i,j} = \prod_{k=1}^{j-1}\frac{q^{i-j+k}-1}{q^k-1} \qquad U_{i,j} = q^{(i-1)(i-2)/2}\prod_{k=1}^{i-1} q^{j-k}-1$$

Combining these produces the following formula for powers of q

$$q^{ij} = \sum_{l=0}^{\min(i,j)}\left(q^{l(l-1)/2}\prod_{k=1}^{l}\frac{(q^{j+1-k}-1)(q^{i-l+k}-1)}{q^k-1}\right)$$

5. Euler and Bernoulli matrices

Euler matrix is a matrix M such that $E_{2n}=(-1)^n(2n)!$ det|M| with n={1,2,3,...}. The explicit form of the matrix M is

$$M_{ij} = \begin{cases} ((2i-2j+2)!)^{-1}, & \text{if i-j+1} \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

For example, when n = 2, the M matrix and its LU decomposition are:

$$\begin{bmatrix} 1/2 & 1 & 0 & 0 & 0 \\ 1/24 & 1/2 & 1 & 0 & 0 \\ \frac{1}{720} & 1/24 & 1/2 & 1 & 0 \\ \frac{1}{40320} & \frac{1}{720} & 1/24 & 1/2 & 1 \\ \frac{1}{3628800} & \frac{1}{40320} & \frac{1}{720} & 1/24 & 1/2 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0 \\ 1/12 & 1 & 0&0&0 \\ \frac{1}{360} & \frac{7}{75} & 1 & 0 & 0 \\ \frac{1}{20160} & \frac{9}{2800} & \frac{323}{3416} & 1 & 0 \\ \frac{1}{1814400} & \frac{11}{189000} & \frac{503}{153720} & \frac{5902}{62325} & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 & 0&0&0 \\ 0 & \frac{5}{12} & 1 & 0 & 0 \\ 0&0&\frac{61}{150} & 1 & 0 \\ 0&0&0&\frac{1385}{3416} & 1 \\ 0&0&0&0&\frac{50521}{124650} \end{bmatrix}$$

The diagonal entries of L and the (i, i+1)[th] entries of U are 1's.
The entries on the diagonal of the U matrix are $U_(ij) = -1/2\frac{euler(2i)}{euler(2i-2)\,i\,(2i-1)}$.

The entries of the L matrix are $L_(ij) = -\frac{1}{euler(2j)}\left(\sum_{k=1}^{j-1}\frac{euler(2k)(2j)!}{(2k)!\,(2i-2k)!} + \frac{(2j)!}{(2i)!}\right)$.

Interestingly, when using this identity for the diagonal terms of the L matrix, where i=j and L_{ij} = 1, we obtained $\sum_{k=1}^{n}\binom{2n}{2k}E_{2k} = 0$,which is a known property of Euler Numbers.

Similarly, ***Bernoulli matrix*** is a matrix N such that $B_{2n} = \frac{-2p!}{2^{2p-2}}$det|N|. The explicit form of matrix N is

$$N_{ij} = \begin{cases} ((2i-2j+3)!)^{-1}, & \text{if i-j+1} \ge 1 \\ 0, & \text{otherwise} \end{cases}.$$

For example when n =5, the N matrix and its LU decomposition are

$$\begin{bmatrix} 1/6 & 1 & 0&0&0 \\ \frac{1}{120} & 1/6 & 1 & 0 & 0 \\ \frac{1}{5040} & \frac{1}{120} & 1/6 & 1 & 0 \\ \frac{1}{362880} & \frac{1}{5040} & \frac{1}{120} & 1/6 & 1 \\ \frac{1}{39916800} & \frac{1}{362880} & \frac{1}{5040} & \frac{1}{120} & 1/6 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0 \\ 1/20 & 1 & 0&0&0 \\ \frac{1}{840} & \frac{3}{49} & 1 & 0&0 \\ \frac{1}{60480} & \frac{11}{7056} & \frac{239}{3720} & 1 & 0 \\ \frac{1}{6652800} & \frac{13}{582120} & \frac{41}{24552} & \frac{818}{12573} & 1 \end{bmatrix} \begin{bmatrix} 1/6 & 1 & 0&0&0 \\ 0 & \frac{7}{60} & 1 & 0&0 \\ 0&0&\frac{31}{294} & 1 & 0 \\ 0&0&0&\frac{127}{1240} & 1 \\ 0&0&0&0&\frac{2555}{25146} \end{bmatrix}.$$

The diagonal entries of L and the (i, i+1)[th] entries of U are always 1's.
The diagonal entries of the U matrix are $U_{jj} = -1/2\frac{bernoulli(2j)(2^{2j}-2)}{bernoulli(2j-2)(2^{2j-2}-2)\,j\,(2j-1)}$

The entries of the L matrix are $L_{ij} = 2\frac{1}{(4^j-2)\,bernoulli(2j)}\left(-\sum_{k=1}^{j-1}\frac{bernoulli(2k)(2j)!\,(2^{2k-1}-1)}{(2k)!\,(2i-2k+1)!} + 1/2\frac{(2j)!}{(2i+1)!}\right)$

Applying this identity of the L matrix to the diagonal terms of L, where i = j and L_{ij}= 1, we obtained $\sum_{k=0}^{j}\frac{\binom{2j}{2k}(2^{2k}-2)bernoulli(2k)}{2j-2k+1} = 0$,which implies a known property of Bernoulli numbers:

$$\sum_{k=0}^{j}(4^k-2)\binom{2j+1}{2k}bernoulli(2k) = 0 .$$

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