Spatial Dependence in Newton-Cartan Gravity in Nonintertial Reference Frames

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Abstract

We study the properties of spatially dependent Newton-Cartan gravity under noninertial, nonrelativistic reference frames. We define the transformation by an element of the set consisting of a rotation matrix which is a continuous function of x and t, a linear transformation function of x and t, and a constant time translation. The set of these transformation elements has the structure of an infinite dimensional semi-group. This semi-group is a generalization of the Galilean line group discussed in [1]. We prove the properties of this semi-group. We calculate the Ricci tensor for this Newtownian spacetime. We calculate the coefficeints in the transformed autoparallel equation, and show all three terms, including the term quadratic in the velocity, are nonzero in the general case. We show how these terms simplify to the autoparallel terms from the previous paper[1] in the case of no spatial dependence.

Introduction

The purpose of this paper is to discuss the group properties of the set of transformations for Cartan's geometric formulation of Newtonian gravity. These transformations are each composed of a rotation matrix, which is a function of x and t, a linear translation function of x and t, and a scalar time translation. For this paper we assume time to be absolute for the full field, and thus the only sensible translations of time are scalar shifts. We will show these translations have the structure of an infinite dimensional semi-group. We will denote this semi-group by $\mathbb{G}*$. We will denote the Galilean line group discussed in [1] as $\mathbb{G}$ for the remainder of the paper. We calculate the transformed auto-parallel equation and the associated coefficients. We then calculate the Ricci tensor based upon this transformed auto-parrallel equation, and show it must remain unchanged. We calculate the torsion tensor for $\mathbb{G}*$.

Formulation of manifolds with an affine connection and gravitation have been analyzed by Cartan [2], Friedrichs [3], and Eisenhart [4]. For studies of Newton-Cartan formulism, see [5], which examines isometries of the manifold, [6], which analyzes the implications to the geometry of the Galilei and Lorentz structures for a four dimensional manifold, [7,8] which investigate Bargmann and conformal structures in the setting of this Newton-Cartan theory, and [9] which exmaines the geometrization of Lagrangain mechanics for nonrelativistic physics. For a paper which reviews all of the previous work on this subject, see [10]. More recently, [11] attempted to develop a general algorithm of diffeomorphism invariance, and [12] studies field theory on Newton-Carton manifolds. Among the important conclusions drawn from these studies, particularly [5-7], is that a certain degenerate contravariant metric and a nowhere vanishing one-form which spans the radical of that metric are key to the Newton-Cartan formalism. However, unlike the relativistic case, there does not exist a unique, torsion-free, affine connection compatible with this metric. The 2-form that results from the non-uniqueness of the connection is subjected to the Duval-Kunzle condition.

Unlike many of the above papers, we use the transformations of coordinate reference frames to describe the manifold, as differential geometry is not generally taught in an undergraduate cirriculum. For an introduction to differential geometry, see [14]. We will use terms such as the affine connection and curvature in the paper, but define them as necessary. This is intended to make our paper more comprehensible to a wider audience.

The motivation for this study is the extension of [1] to a more general case, and an examination of its properties. The transformed autoparallel equation for spacially dependent transformations was expected to exhibit different properties and a different form from the spacially independent transformations. Furthermore, we wanted to examine how the extension would affect the curvature of the Newton-Cartan manifold. These properties could give insight towards the weak-field limit of general relativity.

The paper is organized as follows. In section 2, we define and explain Newton-Cartan gravity and its geometric implementation. In section 3, we review the group $\mathbb{G}$ from the previous paper, and the properties of $\mathbb{G}$. In section 4, we explain and define our semi-group $\mathbb{G}*$more techincally in relation to $\mathbb{G}$. In section 5, we calculate the transformed autoparrallel equation, define the coefficients, and show how these coefficients simplify to the coefficients in [1] under the condition of no spatial dependence. In section 6, we calculate the Ricci tensor and calculate the torsion of our manifold under transformations from $\mathbb{G}*$.

Newton-Cartan Gravity

Newtonian gravity is usually defined in terms of the Universal square law for two masses M and m:

$$F_{gravity} = \frac{GMm}{r^2}, \ (1)$$

with G defined as Newton's gravitational constant. However, we can rewrite this in terms of a gravitational field:

$$\nabla \cdot \vec{g} = -4\pi\rho \ (2),$$

rhere ρ is the mass distribution and we have utilized natural units $G = c = 1$ for convenience, as we will continue to do for the rest of the paper. Here, $\vec{g}$ is the strength of the gravitational field, and the minus sign signifies test particles are pulled toward the center of the mass distribution. Since gravity is path-independent, and thus gravity is conservative, we can sum up Newtonian gravity as:

$$\begin{cases} \nabla \cdot \vec{g} = -4\pi\rho & \text{Gauss' Law} \\ \nabla \times \vec{g} = 0 & \text{gravity is conservative} \end{cases} \quad (3)$$

In Electrodynamics, these properties would allow us to introduce a electric potential. Similarly, we can define $\vec{g} = -\nabla\Phi$, with Φ as the scalar gravitational field potential, such that these equations become The Poisson equation:

$$\nabla^2\Phi = 4\pi\rho \ (4)$$

This equation completely describes the gravitational field due to the mass distribution ρ.

With this equation, we can use Newton's Second Law to describe the motion of a mass m in a gravitational field with potentialΦ:

$$F = m\frac{d\vec{x}^2}{dt^2} = -m\nabla\Phi \ (5)$$

This equation implies the *equivalence principle*, that is, that the test particle couples to the gravitational field by its (inertial) mass. Thus, the masses drop out, and the acceleration of any particle in this field is $\vec{g}$, and thus all particles have the same trajectory in this field.

Following Einstein's publication of General Relativity (GR), Cartan [2] showed that Newtonian gravity can, as in Einstein's general relativity, be recast as a geometric theory; see also [3], [4], and [10]. As in GR, the equivelence principle is central to this formulation of the theory, such that any solution to (5) can be described as an autoparrallel curve. Let us examine universal (Galilean) time $\tau = \lambda t + b$, where λ and b are constants. Then let us rewrite (5) as

$$0 = \frac{d^2x^i}{d\tau^2} + \delta^{ij}\frac{\partial\Phi}{\partial x^j}\left(\frac{dt}{d\tau}\right)^2, \ (6)$$

where we have adopted Einstein summation notation and δ^{ij} is the Euclidean metric used here to raise/lower indices. We will follow these conventional notations for the remainder of the paper. Note all Latin indices run over 1, 2, and 3.

In order to establish Cartan's geometric theory, we recognize that (6) is of the same form as the equation that describes how tangent vectors on a manifold are related. In a general manifold, it is impossible to ask if two vectors at different points are parrallel. However, the *affine connection* is a basic geometric notion which allows us to relate two vectors at different points of a manifold. The components of the affine connection are conventionally written as $\Gamma^{\mu}_{\alpha\beta}$. Note Greek indices run over 0, 1, 2, and 3. [15] provide more detailed discussions of the affine connection. These connections lead to the auto-parrallel equation:

$$\frac{d^2x^\mu}{d\tau} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, \ (7)$$

where $\Gamma^{\mu}_{\beta\alpha}$ is called the connection coefficients. This is comparable to the Euler-Lagrange equations applied geometrically to a manifold with the connection. Comparing (6) and (7) shows that (6) describes an auto-parallel curve in which the only nonzero connection coefficients are:

$$\Gamma^{\mu}_{00} = \delta^{ij}\frac{\partial\Phi}{\partial x^j} = -\vec{g} \ (8)$$

As these coefficients do not all vanish, in Newtonain gravity, as in GR, matter introduces a curvature into the gravitational field. Furthermore, the trajectory of a particle under the influence of Newtonian gravity can be described as an auto-parallel curve in this curved spacetime.

The Ricci tensor, which describes the curvature of the spacetime, is calculated in this field to be:

$$R_{\alpha\beta} = \partial_\rho\Gamma^{\rho}_{\alpha\beta} - \partial_\beta\Gamma^{\rho}_{\rho\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} \ (9)$$

and has one nonzero component:

$$R_{00} = \partial_l\Gamma^{l}_{00} = \partial_l\partial^l\Phi \ (10)$$

Combining this equation for the Ricci tensor with Gauss' Law $\partial_i g^i = 4\pi\rho$, we obtain the relationship between the matter density ρ and the curvature of the Newton-Cartan spacetime:

$$-\partial_i g^i = R_{00} = 4\pi\rho \ (11)$$

Note that ρ can be defined/calculated in two ways: through Gauss' Law (3) and through the curvature of the spacetime (11). (11) here beomces the Newton-Cartan analogue for the field equations in Einstein's GR.

Note also that Newton-Cartan spacetime differs greatly from the spacetime of GR in that Newton-Cartan spaceime lacks a metric. Thus, (7) is defined as an auto-parrallel equation, rather than a geodesic equation. Note also that the connection coefficients need not be symmetric in the lower indices, although only the symmetric component of these coefficients enters the auto-parrallel equation.

Galilean Line Group $\mathbb{G}$

In this section we introduce the Galilean and Galilean Line group from the previous paper, and explain how these groups transform within Newton-Cartan theory. The Galilean group $\mathcal{G}$ is defined by elements of the form,

$$\mathcal{G} = \{(A, \mathbf{v}, \mathbf{a}, b)\}\,, \ (12)$$

where A is a rotation matrix, $\mathbf{v}$ is a velocity boost between frames, $\mathbf{a}$ is the spatial translation. and b is the time translation. These 4 elements can completely describe the differences between any two arbitrary reference frames within Galilean spacetime. We define two elements within this group to compose as:

[

$$(A_2, \mathbf{v}_2, \mathbf{a}_2, b_2)\,(A_1, \mathbf{v}_1, \mathbf{a}_1, b_1) = (A_2A_1, \mathbf{v}_2 + A_2\mathbf{v}_1, \mathbf{a}_2 + A_2\mathbf{a}_1 + b_1\mathbf{v}_2, b_2 + b_1)\,. \ (13)$$

Then, we can show that this set is associative if $[(A_3, \mathbf{v}_3, \mathbf{a}_3, b_3)\,(A_2, \mathbf{v}_2, \mathbf{a}_2, b_2)]\,(A_1, \mathbf{v}_1, \mathbf{a}_1, b_1) = (A_3, \mathbf{v}_3, \mathbf{a}_3, b_3)\,[(A_2, \mathbf{v}_2, \mathbf{a}_2, b_2)\,(A_1, \mathbf{v}_1, \mathbf{a}_1, b_1)]$.

$$\begin{aligned}[(A_3, \mathbf{v}_3, \mathbf{a}_3, b_3)\,(A_2, \mathbf{v_2}, \mathbf{a_2}, b_2)]\,(A_1, \mathbf{v_1}, \mathbf{a_1}, b_1) &= (A_3A_2, \mathbf{v}_3 + A_3\mathbf{v}_2, \mathbf{a}_3 + A_3\mathbf{a}_2 + b_2\mathbf{v}_3, b_3 + b_2) \\ &\times \ (A_1, \mathbf{v}_1, \mathbf{a}_1, b_1) \ (13b)\end{aligned}$$

]

It can be shown that this set is associative and the identity is $(\mathbb{I}, 0, 0, 0)$, where $\mathbb{I}$ is the identity matrix. Furthermore, it can be shown each element has an inverse of the form:

$$(A, \mathbf{v}, \mathbf{a}, b)^{-1} = \left(A^{-1}, -A^{-1}\mathbf{v}, -A^{-1}(\mathbf{a} - b\mathbf{v}), -b\right), \ (14)$$

which must exist because any rotation matrix has an inverse. Finally, this set is trivially closed, because rotation matrix multiplication is closed. Therefore, this set must be a group, referred to as the Galilean group.

With this group, the action of transformation on any spacetime point $(\mathbf{x}, t)$can be given as:

$$(A, \mathbf{v}, \mathbf{a}, b) : \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{x}' \\ t' \end{pmatrix} = \begin{pmatrix} A\mathbf{x} + \mathbf{v}t + \mathbf{a} \\ t + b \end{pmatrix} \quad (15)$$

This group of transformations is satisfactory to describe all transformations between inertial reference frames. This group is isomorphic to a subgroup of $\mathbb{G}$, which describes transformations between all non-inertial (and inertial) reference frames. An element is given by the form $(A(t), \mathbf{a}(t), b)$.

It acts on a spacetime point as:

$$(A(t), \mathbf{a}(t), b) : \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{x}' \\ t' \end{pmatrix} = \begin{pmatrix} A(t)\mathbf{x} + \mathbf{a}(t) \\ t + b \end{pmatrix} \quad (16)$$

Note that in this group, information concerning the velocity boost can be contained within the spatial translation element because of the time dependence (i.e., $\mathbf{a}(t) = \mathbf{v}t + \mathbf{a}$ will act identically in this group as the two elements $\mathbf{v}$ and $\mathbf{a}$ in the Galilean group). This shows the isomorphism to a subgroup, because elements in $\mathbb{G}$of the form $(A, \mathbf{v}t + \mathbf{a}, b)$, where A, $\mathbf{v}$, and $\mathbf{a}$ are not dependent on t, act in the same way on an arbitrary spacetime element as an element of $\mathcal{G}$.

From this, we deduce the composition rule for two elements in this group:

$$(A_2, \mathbf{a}_2, b_2)(A_1, \mathbf{a}_1, b_1) = ((\Lambda_{b_1} A_2) A_1, (\Lambda_{b_1} A_2) \mathbf{a}_1 + \Lambda_{b_1} \mathbf{a}_2, b_2 + b_1), \quad (17)$$

Where Λ_a is called the shift operator and acts on a function by shifting its argument by a: $\Lambda_b f(t) = f(t + b)$. Thus

$$\Lambda_{b_1} A_2(t) = A_2(t + b_1), \quad (18)$$

It can now be shown $\mathbb{G}$ is a group. We can first show that the identity is the element $(\mathbb{I}, 0, 0)$ by having it act on an abritrary element of the group:

$$\begin{aligned} (\mathbb{I}, 0, 0)(A, \mathbf{a}, b) &= (\mathbb{I}A, \mathbb{I}\mathbf{a} + 0, b) \\ &= (A, \mathbf{a}, b) \\ (A, \mathbf{a}, b)(\mathbb{I}, 0, 0) &= (\Lambda_0 A\mathbb{I}, \Lambda_0 A \cdot 0 + \Lambda_0 \mathbf{a}, b) \\ &= (A, \mathbf{a}, b) \quad (19) \end{aligned}$$

Therefore, $(\mathbb{I}, \mathbf{a}, b)$ is the (right and left) identity.

Next, we must show that each element has an inverse. We claim that the inverse of each element $(A, \mathbf{a}, b)$ is the element $\left(\Lambda_{-b} A^{-1}, -\Lambda_{-b}\left(A^{-1}\mathbf{a}\right), -b\right)$.

$$
\begin{aligned}
(A,\mathbf{a},b)\left(\Lambda_{-b}A^{-1},-\Lambda_{-b}\left(A^{-1}\mathbf{a}\right),-b\right) &= \left(\left(\Lambda_{-b}A\right)\Lambda_{-b}A^{-1},\left(\Lambda_{-b}A\right)\left(-\Lambda_{-b}\left(A^{-1}\mathbf{a}\right)\right)+\Lambda_{-b}\mathbf{a},b-b\right)\\
&= \left(\mathbb{I},-\Lambda_{-b}\mathbf{a}+\Lambda_{-b}\mathbf{a},0\right)\\
&= \left(\mathbb{I},0,0\right)\\
\left(\Lambda_{-b}A^{-1},-\Lambda_{-b}\left(A^{-1}\mathbf{a}\right),-b\right)(A,\mathbf{a},b) &= \left(\left(\Lambda_{b}\Lambda_{-b}A^{-1}\right)A,\left(\Lambda_{b}\Lambda_{-b}A^{-1}\right)\mathbf{a}+\Lambda_{b}-\Lambda_{-b}\left(A^{-1}\mathbf{a}\right),-b+b\right)\\
&= \left(\mathbb{I},A^{-1}\mathbf{a}-A^{-1}\mathbf{a},0\right)\\
&= \left(\mathbb{I},0,0\right)\ (20)
\end{aligned}
$$

Therefore, the element $\left(\Lambda_{-b}A^{-1},-\Lambda_{-b}\left(A^{-1}\mathbf{a}\right),-b\right)$ is the left and right inverse of the an arbitrary element $(A,\mathbf{a},b)$. This group is similarly trivially closed, with the same justification as that for the previous group. We must now show associativity in this group. This group will be associative if $((A_3,\mathbf{a}_3,b_3)(A_2,\mathbf{a}_2,b_2))(A_1,\mathbf{a}_1,b_1)=(A_3,\mathbf{a}_3,b_3)((A_2,\mathbf{a}_2,b_2)(A_1,\mathbf{a}_1,b_1))$.

$$
\begin{aligned}
((A_3,\mathbf{a}_3,b_3)(A_2,\mathbf{a}_2,b_2))(A_1,\mathbf{a}_1,b_1) &= \left(\left(\Lambda_{b_2}A_3\right)A_2,\left(\Lambda_{b_2}A_3\right)\mathbf{a}_2+\Lambda_{b_2}\mathbf{a}_3,b_3+b_2\right)(A_1,\mathbf{a}_1,b_1)\\
&= \left(\left(\Lambda_{b_1}\left(\left(\Lambda_{b_2}A_3\right)A_2\right)\right)A_1,\left(\Lambda_{b_1}\left(\Lambda_{b_2}A_3\right)A_2\right)\mathbf{a}_1\right.\\
&+ \left.\Lambda_{b_1}\left(\left(\Lambda_{b_2}A_3\right)\mathbf{a}_2+\Lambda_{b_2}\mathbf{a}_3\right),b_3+b_2+b_1\right)\\
&= \left(\Lambda_{b_2+b_1}A_3\Lambda_{b_2}A_2A_1,\left(\Lambda_{b_2+b_1}A_3\Lambda_{b_1}A_2\right)\mathbf{a}_1\right.\\
&+ \left.\left(\Lambda_{b_2+b_1}A_3\right)\Lambda_{b_1}\mathbf{a}_2+\Lambda_{b_2+b_1}\mathbf{a}_3,b_3+b_2+b_1)\right)\\
(A_3,\mathbf{a}_3,b_3)((A_2,\mathbf{a}_2,b_2)(A_1,\mathbf{a}_1,b_1)) &= (A_3,\mathbf{a}_3,b_3)\left(\left(\Lambda_{b_1}A_2\right)A_1,\left(\Lambda_{b_1}A_2\right)\mathbf{a}_1+\Lambda_{b_1}\mathbf{a}_2,b_2+b_1\right)\\
&= \left(\left(\Lambda_{b_2+b_1}A_3\right)\left(\Lambda_{b_1}A_2\right)A_1,\left(\Lambda_{b_2+b_1}A_3\right)\right.\\
&\times \left.\left(\left(\Lambda_{b_1}A_2\right)\mathbf{a}_1+\Lambda_{b_1}\mathbf{a}_2\right)+\Lambda_{b_2+b_1}\mathbf{a}_3,b_3+b_2+b_1\right)\\
&= \left(\Lambda_{b_2+b_1}A_3\Lambda_{b_2}A_2A_1,\left(\Lambda_{b_2+b_1}A_3\Lambda_{b_1}A_2\right)\mathbf{a}_1\right.\\
&= \left.\left(\Lambda_{b_2+b_1}A_3\right)\Lambda_{b_1}\mathbf{a}_2+\Lambda_{b_2+b_1}\mathbf{a}_3,b_3+b_2+b_1\right)\ (21)
\end{aligned}
$$

Therefore, the $\mathbb{G}$ is associative, and must therefore be a group. This group has been called [1] the Galilean Line group, $\mathbb{G}$. A more complete analysis of this group may be found in [13].

$\mathbb{G}*$, the Extension of $\mathbb{G}$

We now examine the group $\mathbb{G}^*$, whose behavior we examine in the remainder of this paper. We begin by showing this set is associative, has an identity, and is closed, but not all elements have inverses. Thus, we will show $\mathbb{G}^*$ is a semi-group by definition.

The elements of $\mathbb{G}^*$are defined to be of the form $(A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b)$. For a spacetime point $(\mathbf{x},t)$, this element operates on the point to transform it to $(\mathbf{x}',t)$ as follows:

$$
(A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b):\begin{pmatrix}\mathbf{x}\\ t\end{pmatrix}\rightarrow\begin{pmatrix}\mathbf{x}'\\ t'\end{pmatrix}=\begin{pmatrix}A(\mathbf{x},t)\mathbf{x}+\mathbf{a}(\mathbf{x},t)\\ t+b\end{pmatrix}\ (22)
$$

Note $\mathbb{G}$ is clearly a subgroup of $\mathbb{G}^*$, as any element in $\mathbb{G}$ must be in $\mathbb{G}^*$ and will act the same as it did in $\mathbb{G}$ with no spatial dependence. It follows that two elements compose as follows:

$$\begin{aligned}(A_2(\mathbf{x},t),\mathbf{a}_2(\mathbf{x},t),b_2)(A_1(\mathbf{x},t),\mathbf{a}_1(\mathbf{x},t),b_1) &= (\Lambda_{A_1\mathbf{x}+\mathbf{a}}\Lambda_{b_1}A_2)A_1,(\Lambda_{A_1\mathbf{x}+\mathbf{a}}\Lambda_{b_1}A_2)\mathbf{a}_1\\ &+ +\Lambda_{A_1\mathbf{x}+\mathbf{a}}\Lambda_{b_1}\mathbf{a}_2, b_2+b_1, \quad (23)\end{aligned}$$

where $\Lambda_{A_1\mathbf{x}+\mathbf{a_1}}\Lambda_{b_1}A_2(\mathbf{x},t) = A_2(A_1\mathbf{x}+\mathbf{a}_1, t+b_1)$, which follows from the following definition of the operations of the composition of $f_2 = (A_2(\mathbf{x},t),\mathbf{a}_2(\mathbf{x},t),b_2)$ and $f_1 = (A_1(\mathbf{x},t),\mathbf{a}_1(\mathbf{x},t),b_1)$: $f_2 \circ f_1$:

$$\begin{aligned}f_2\circ f_1 : \begin{pmatrix}\mathbf{x}\\ t\end{pmatrix} \quad \to \quad \begin{pmatrix}\mathbf{x}''\\ t''\end{pmatrix} &= \begin{pmatrix}A_2(\mathbf{x}(f_1),t+b_1)\mathbf{x}(f_1)+\mathbf{a}_2(\mathbf{x}(f_1),t+b_1)\\ t+b_2+b_1\end{pmatrix}\\ &= \begin{pmatrix}(\Lambda_{A_1\mathbf{x}+\mathbf{a_1}}\Lambda_{b_1}A_2)\mathbf{x}'+\Lambda_{A_1\mathbf{x}+\mathbf{a_1}}\Lambda_{b_1}\mathbf{a}_2\\ t+b_2+b_1\end{pmatrix}, \quad (24)\end{aligned}$$

where $\mathbf{x}'$ is as in (22). Then we will now show associativity within $\mathbb{G}^*$. Note that in proofs, we will forsake Λ notation for clarity, and write out each term in full.

We want to show the operation $\circ$ is accociative, with $\circ$ defined as the composition of two group elements, as below. Then we must show:

$$f_3\circ(f_2\circ f_1) \quad = (f_3\circ f_2)\circ f_1 \quad (25)$$

, where f_1, f_2, f_3 are in our group and

$$\begin{aligned}f_1 &= (A_1,\mathbf{a_1},b_1)\\ f_2 &= (A_2,\mathbf{a}_2,b_2)\\ f_3 &= (A_3,\mathbf{a}_3,b_3)\end{aligned}$$

Let

$$t_1 = t+b_1 \qquad t_2 = t+b_1+b_2 \qquad t_3 = t+b_1+b_2+b_3$$

$$t_4 = t+b_2+b_3$$

$$\begin{aligned}\mathbf{x}(f_1) &= A_1(\mathbf{x},t_1)\mathbf{x}+\mathbf{a}_1(\mathbf{x},t_1)\\ \mathbf{x}(f_2) &= A_2(\mathbf{x},t_2)\mathbf{x}+\mathbf{a}_2(\mathbf{x},t_2)\\ \mathbf{x}(f_3) &= A_3(\mathbf{x},t_3)x+\mathbf{a}_3(\mathbf{x},t_3)\end{aligned}$$

Then

$$
\begin{aligned}
\mathbf{x}\left(f_2 \circ f_1\right) &= A_2\left(\mathbf{x}\left(f_1\right), t_2\right) \mathbf{x}\left(f_1\right) + a_2\left(\mathbf{x}\left(f_1\right), t_2\right) \quad (26) \\
&= A_2\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right)\left(A_1(\mathbf{x}, t_1)\mathbf{x} + a_1(\mathbf{x}, t_1) + a_2\left(A_1(\mathbf{x}, t_1)\mathbf{x} + a_1(\mathbf{x}, t_1), t_2\right)\right. \\
\mathbf{x}\left(f_3 \circ f_2\right) &= A_3\left(\mathbf{x}\left(f_2\right), t_4\right) \mathbf{x}\left(f_2\right) + \mathbf{a}_3\left(\mathbf{x}\left(f_2\right), t_4\right) \\
&= A_3\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + a_2(\mathbf{x}, t_2), t_4\right)\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + \mathbf{a}_2(\mathbf{x}, t_2)\right) \\
&+ \mathbf{a}_3\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + \mathbf{a}_2(\mathbf{x}, t_2), t_4\right) (27) \\
\mathbf{x}\left(f_3 \circ \left(f_2 \circ f_1\right)\right) &= \mathbf{x}\left(A_3\left(\mathbf{x}\left(f_2 \circ f_1\right), t_3\right)\left(\mathbf{x}\left(f_2 \circ f_1\right)\right) + \mathbf{a}_3\left(\mathbf{x}\left(f_2 \circ f_1\right), t_3\right)\right) \\
&= A_3[A_2\left[A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right]\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1)\right) \\
&+ \mathbf{a}_2(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2), t_3] \\
&\times \left(A_2\left[\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1)\right), t_2\right]\right) \\
&\times \left(A_1\left(\mathbf{x}, t_1\right)\mathbf{x} + \mathbf{a}_1\left(\mathbf{x}, t_1\right) + \mathbf{a}_2\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right)\right) \\
&+ \mathbf{a}_3[A_2\left[A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right]\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1)\right) \\
&+ \mathbf{a}_2\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right), t_3] \ (28) \\
\mathbf{x}\left(\left(f_3 \circ f_2\right) \circ f_1\right) &= \mathbf{x}(A_3\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + \mathbf{a}_2(\mathbf{x}, t_2), t_4\right)\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + \mathbf{a}_2(\mathbf{x}, t_2)\right) \\
&+ \mathbf{a}_3\left(A_2\left(\mathbf{x}, t_2\right)\mathbf{x} + \mathbf{a}_2(\mathbf{x}, t_2), t_4\right) \circ f_1) \\
&= A_3\left(A_2\left(\mathbf{x}(f_1), t_2\right)\mathbf{x}(A_1) + \mathbf{a}_2(\mathbf{x}(f_1), t_2), t_3\right)\left(A_2\left(\mathbf{x}(f_1), t_2\right)\mathbf{x}(f_1) + a_2(\mathbf{x}(f_1), t_2)\right) \\
&+ \mathbf{a}_3\left(A_2\left(\mathbf{x}(f_1), t_2\right)\mathbf{x}(A_1) + \mathbf{a}_2(\mathbf{x}(f_1), t_2), t_3\right) \\
&= A_3[A_2\left[A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right]\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1)\right) \\
&+ \mathbf{a}_2(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2), t_3] \\
&\times \left(A_2\left[\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1)\right), t_2\right]\right) \\
&\quad \left(A_1\left(\mathbf{x}, t_1\right)\mathbf{x} + \mathbf{a}_1\left(\mathbf{x}, t_1\right) + \mathbf{a}_2\left(A_1(\mathbf{x}, t_1)\mathbf{x} + \mathbf{a}_1(\mathbf{x}, t_1), t_2\right)\right) \ (29)
\end{aligned}
$$

Since these expressions are equal, $\mathbf{x}\left(f_3 \circ \left(f_2 \circ f_1\right)\right) = \mathbf{x}\left(\left(f_3 \circ f_2\right) \circ f_1\right)$, and thus this operation is associative for x. Associativity for the time element is straightforward:

$$
\begin{aligned}
t(f_2 \circ f_1) &= t + b_2 + b_1 \\
t\left(f_3 \circ f_2\right) &= t + b_3 + b_2 \\
t(f_3 \circ \left(f_2 \circ f_1\right)) &= b_3 + \left(t + b_2 + b_1\right) \\
t(\left(f_3 \circ f_3\right) \circ f_1) &= \left(t + b_3 + b_2\right) + b_1
\end{aligned}
$$

As $t(f_3 \circ (f_2 \circ f_1)) = t((f_3 \circ f_2) \circ f_1)$, the time element must also therefore be associative.

Therefore, the composition operation of $\mathbb{G}^*$ is associative for all elements of the set.

Next, we must show that the element $(\mathbb{I}, 0, 0)$ is the left and right identity of $\mathbb{G}^*$. Take an arbitrary element of $\mathbb{G}^*$, $(A(\mathbf{x}, t), \mathbf{a}(\mathbf{x}, t), b)$.

$$
\begin{aligned}
(\mathbb{I},0,0)\,(A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b) &= (\mathbb{I}A(\mathbf{x},t),\mathbb{I}\cdot\mathbf{a}(\mathbf{x},t)+\mathbb{I}\cdot 0,0+b)\\
&= (A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b)\ \ (30)\\
(A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b)\,(\mathbb{I},0,0) &= ((\Lambda_{\mathbb{I}\cdot\mathbf{x}}\Lambda_0 A(\mathbf{x},t))\,\mathbb{I},(\Lambda_{\mathbb{I}\cdot\mathbf{x}}\Lambda_0 A(\mathbf{x},t))\cdot 0+\Lambda_{\mathbb{I}\cdot\mathbf{x}}\Lambda_0\mathbf{a}(\mathbf{x},t),b+0)\\
&= (A(\mathbf{x},t),\mathbf{a}(\mathbf{x},t),b)\ \ (31)
\end{aligned}
$$

Therefore, $(\mathbb{I},0,0)$ is the unique identity of $\mathbb{G}^*$. The closure of $\mathbb{G}^*$ is trivial in the same way as $\mathbb{G}$, since the $\Lambda_{f(\mathbf{x})}$ operation will not change the form of the elements of $\mathbb{G}^*$.

We will now show that there exists an element of $\mathbb{G}^*$does not have an inverse. One element in our set is $A(\mathbf{x},t)=\mathbb{I}$ and $a(\mathbf{x},t)=a(\mathbf{x})$, which yields the spacial transformation equation of form:

$$x'^j = x^j + a^j(x^j)\ \ (32)$$

As an element of our set, this must have an inverse for our set to be a group. I will show this element does not have an inverse, and thus our set cannot be a group.

We need to find a function in our set, the inverse of $\mathbf{a}(\mathbf{x})$ in this set, call it $\mathbf{g}(\mathbf{x}')$, such that

$$g(x'^j) = g(x^j + a^j) = x^j\ \ (33),$$

In which $\mathbf{g}(\mathbf{x}')$, transforming after $\mathbf{a}(\mathbf{x})$ has transformed the function, must be dependent upon $\mathbf{x}'$.

Take the MacLaurin series of $g(x_2^j)$:

$$g(x'^j) = \Sigma_{n=0}^{\infty}\frac{g^{(n)}(0)(x^j+a^j)^n}{n!}\ \ (34)$$

But clearly, these two equations are never going to be equal, as the $g^{(n)}(0)(x^j + (a^j(x^j))^n$ will not simplify to x^j, since $g^{(n)}(0)$ must be a constant. Therefore, no element in our set of this form has an inverse, and thus our set cannot be a group by definition. Having proven associativity, an identity, and closure, the set can be reffered to as a semi-group, terminology we will adopt for the rest of the paper.

Transformed Autoparrallel Equation

We now seek to transform the autoparrallel equation.

We begin with the autoparallel equation for the untransformed frame, (6):

$$\frac{d^2x^j(x,t)}{d\tau^2} + \delta^{ij}\frac{\partial x'^k}{\partial x^i}\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 = 0(6)$$

We can transform (6) to another frame under the understanding that x is a function of x' and t'. From (22), we know:

$$x' = A(x,t)\vec{x} + \vec{a}(x,t) \implies x^k = (A_j)^k (x,t)(x'^j - a^j(x,t)) \quad (35)$$

We want an autoparallel of the form:

$$\frac{d^2 x'^j}{d\tau^2} + \Gamma^j_{\alpha\beta}\frac{dx'^\alpha}{d\tau}\frac{dx'^\beta}{d\tau} = 0(7b)$$

We then calculate the first derivative and second derivative of $x^j(x,t)$ with respect to τ. Then

$$\begin{aligned}
\frac{dx^k}{d\tau} &= (A_j)^k (x,t)(\frac{dx'^j}{dt}\frac{dt}{d\tau} - \frac{da^j}{dt}\frac{dt}{d\tau} - \frac{da^j}{dx^l}\frac{dx^l}{d\tau}) \\
&+ \left(\frac{d\,(A_j)^k}{dt}\frac{dt}{d\tau} + \frac{d\,(A_j)^k}{dx^l}\frac{dx^l}{d\tau}\right)(x'^j - a^j) \\
&= (A_j)^k (\frac{dx'^j}{dt}\frac{dt}{d\tau} - \dot{a}^j(x,t)\frac{dt}{d\tau} - \frac{da^j}{dx^l}\frac{dx^l}{d\tau}) \\
&+ \left(\left(\dot{A}_j\right)^k\frac{dt}{d\tau} + \frac{d\,(A_j)^k}{dx^l}\frac{dx^l}{d\tau}\right)(x'^j - a^j) \quad (36)
\end{aligned}$$

where $\dot{p} = \frac{dp}{dt}$ for any function (or matrix of functions) p.

Also note $\frac{\partial x'^k}{\partial x^i} = (A_j)^k$. Then the first term in the autoparallel equation transforms as:

$$\begin{aligned}
\frac{d^2 x^k}{d\tau^2} &= (A_j)^k (\frac{d^2 x'^j}{dt^2}\left(\frac{dt}{d\tau}\right)^2 - \ddot{a}^j\left(\frac{dt}{d\tau}\right)^2 - 2\frac{d\dot{a}^j}{dx^l}\frac{dx^l}{d\tau}\frac{dt}{d\tau} - \frac{d^2 a^j}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}) \\
&+ \left(\left(\dot{A}_j\right)^k\frac{dt}{d\tau} + \frac{d\,(A_j)^k}{dx'^l}\frac{dx'^l}{d\tau}\right)(\frac{dx'^j}{d\tau} - \dot{a}^j\frac{dt}{d\tau} - \frac{da^j}{dx'^m}\frac{dx'^m}{d\tau}) \\
&+ \left(\left(\dot{A}_j\right)^k\frac{dt}{d\tau} + \frac{d\,(A_j)^k}{dx'^l}\frac{dx'^l}{d\tau}\right)(\frac{dx'^j}{d\tau} - \dot{a}^j\frac{dt}{d\tau} - \frac{da^j}{dx^m}\frac{dx^m}{d\tau}) \\
&+ \left(\left(\ddot{A}_j\right)^k\left(\frac{dt}{d\tau}\right)^2 + 2\frac{d\left(\dot{A}_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + \frac{d^2\,(A_j)^k}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right)(x'^j - a^j) \quad (37)
\end{aligned}$$

The second term reads:

$$
\begin{aligned}
\delta^{ij}\frac{\partial x'^k}{\partial x^i}\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 &= \delta^{ij}\frac{\partial\left(\left(A_g\right)^k x^g + a^k\right)}{\partial x^i}\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 \\
\delta^{ij}\frac{\partial\left(\left(A_g\right)^k x^g + a^k\right)}{\partial x^i} &= \delta^{ij}\left(A_g\right)^k\frac{\partial x^g}{\partial x^i} + \delta^{ij}\frac{\partial\left(A_g\right)^k}{\partial x^i}x^g + \delta^{ij}\frac{\partial a^k}{\partial x^i} \\
\delta^{ij}\frac{\partial x'^k}{\partial x^i}\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 &= \left(\delta^{ij}\left(A_g\right)^k\frac{\partial x'^g}{\partial x^i} + \delta^{ij}\frac{\partial\left(A_g\right)^k}{\partial x^i}x'^g + \delta^{ij}\frac{\partial a^k}{\partial x^i}\right)\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2
\end{aligned} \tag{38}
$$

Then the autoparallel becomes:

$$
\begin{aligned}
0 &= \frac{d^2x^j(x,t)}{d\tau^2} + \delta^{ij}\frac{\partial x'^k}{\partial x^i}\frac{\partial\Phi(x(x',t'))}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 \\
&= \left(A_j\right)^k\left(\frac{d^2x'^j}{dt^2}\left(\frac{dt}{d\tau}\right)^2 - \ddot{a}^j\left(\frac{dt}{d\tau}\right)^2 - 2\frac{d\dot{a}^j}{dx^l}\frac{dx^l}{d\tau}\frac{dt}{d\tau} - \frac{d^2a^j}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
&+ 2\left(\left(\dot{A}_j\right)^k\frac{dt}{d\tau} + \frac{d\left(A_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\right)\left(\frac{dx'^j}{d\tau} - \dot{a}^j\frac{dt}{d\tau} - \frac{da^j}{dx'^m}\frac{dx'^m}{d\tau}\right) \\
&+ \left(\left(\ddot{A}_j\right)^k\left(\frac{dt}{d\tau}\right)^2 + 2\frac{d\left(\dot{A}_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right)\left(x'^j - a^j\right) \\
&+ \left(\delta^{ij}\left(A_g\right)^k\frac{\partial x'^g}{\partial x^i} + \delta^{ij}\frac{\partial\left(A_g\right)^k}{\partial x^i}x'^g + \delta^{ij}\frac{\partial a^k}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 \quad (39)
\end{aligned}
$$

Expand terms:

$$
\begin{aligned}
0 = {} & (A_j)^k \Big(\frac{d^2x'^j}{dt^2}\left(\frac{dt}{d\tau}\right)^2 - \ddot{a}^j\left(\frac{dt}{d\tau}\right)^2 - 2\frac{d\dot{a}^j}{dx^l}\frac{dx^l}{d\tau}\frac{dt}{d\tau} - \frac{d^2a^j}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\Big) \\
& + 2\left(\left(\dot{A}_j\right)^k\frac{dx'^j}{d\tau}\frac{dt}{d\tau} - \left(\dot{A}_j\right)^k\dot{a}^j\frac{dt}{d\tau} - \left(\dot{A}_j\right)^k\frac{da^j}{dx'^m}\frac{dx'^m}{d\tau}\frac{dt}{d\tau}\right) \\
& + 2\left(\frac{d\left(A_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\frac{dx'^j}{d\tau} - \frac{d\left(A_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\dot{a}^j\frac{dt}{d\tau} - \frac{d\left(A_j\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\frac{da^j}{dx'^m}\frac{dx'^m}{d\tau}\right) \\
& + \left(\left(\ddot{A}_j\right)^k x'^j\left(\frac{dt}{d\tau}\right)^2 + 2\frac{d\left(\dot{A}_j\right)^k}{dx'^l}x'^j\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}x'^j\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
& - \left(\left(\ddot{A}_j\right)^k a^j\left(\frac{dt}{d\tau}\right)^2 + 2\frac{d\left(\dot{A}_j\right)^k}{dx'^l}a^j\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}a^j\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
& + \left(\delta^{ik}\left(A_g\right)^p\frac{\partial x'^g}{\partial x^i} + \delta^{ik}\frac{\partial\left(A_g\right)^p}{\partial x^i}x'^g + \delta^{ik}\frac{\partial a^p}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^p}\left(\frac{dt}{d\tau}\right)^2 \quad (40)
\end{aligned}
$$

To get it in the form of (6), we want to multiply this equation on the left by the matrix $\left((A_j)^k\right)^T = (A_k)^n$, the inverse of $(A_j)^k$:

$$
\begin{aligned}
0 = {} & \frac{d^2x'^n}{dt^2}\left(\frac{dt}{d\tau}\right)^2 - \ddot{a}^n\left(\frac{dt}{d\tau}\right)^2 - 2\frac{d\dot{a}^n}{dx^l}\frac{dx^l}{d\tau}\frac{dt}{d\tau} - \frac{d^2a^n}{dx'^l dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau} \\
& + 2\left((A_k)^n\left(\dot{A}_j\right)^k\frac{dx'^j}{d\tau}\frac{dt}{d\tau} - (A_k)^n\left(\dot{A}_j\right)^k\dot{a}^j\frac{dt}{d\tau} - (A_k)^n\left(\dot{A}_j\right)^k\frac{da^j}{dx'^m}\frac{dx'^m}{d\tau}\frac{dt}{d\tau}\right) \\
& + 2\left((A_k)^n\frac{d\left(A_m\right)^k}{dx'^l}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau} - (A_k)^n\frac{d\left(A_j\right)^k}{dx'^l}\dot{a}^j\frac{dx'^l}{d\tau}\frac{dt}{d\tau} - (A_k)^n\frac{d\left(A_j\right)^k}{dx'^l}\frac{da^j}{dx'^m}\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
& + \left((A_k)^n\left(\ddot{A}_j\right)^k x'^j\left(\frac{dt}{d\tau}\right)^2 + 2\left(A_k\right)^n\frac{d\left(\dot{A}_j\right)^k}{dx'^l}x'^j\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + (A_k)^n\frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}x'^j\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
& - \left((A_k)^n\left(\ddot{A}_j\right)^k a^j\left(\frac{dt}{d\tau}\right)^2 + 2\left(A_k\right)^n\frac{d\left(\dot{A}_j\right)^k}{dx'^l}a^j\frac{dx'^l}{d\tau}\frac{dt}{d\tau} + (A_k)^n\frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}a^j\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau}\right) \\
& + \left(\delta^{in}\frac{\partial x'^k}{\partial x^i} + \delta^{ik}\left(A_p\right)^n\frac{\partial\left(A_g\right)^p}{\partial x^i}x'^g + \delta^{ik}\frac{\partial a^n}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k}\left(\frac{dt}{d\tau}\right)^2 \quad (41)
\end{aligned}
$$

Then the coefficients must be:

$$\Gamma'^{n}_{lm} = -\frac{d^2a^n}{dx'^l dx'^m} - 2\left(A_k\right)^n \frac{d\left(A_j\right)^k}{dx'^l}\frac{da^j}{dx'^m} - \left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}a^j$$

$$+\left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}x'^j + 2\left(A_k\right)^n \frac{d\left(A_m\right)^k}{dx'^l} \quad (42)$$

$$\Gamma'^{n}_{00} = -\ddot{a}^n - 2\left(A_k\right)^n \left(\dot{A}_j\right)^k \dot{a}^j - \left(A_k\right)^n \left(\ddot{A}_j\right)^k a^j + \left(A_k\right)^n \left(\ddot{A}_j\right)^k x'^j$$

$$+\left(\delta^{in}\frac{\partial x'^k}{\partial x^i} + \delta^{ik}\left(A_p\right)^n \frac{\partial\left(A_g\right)^p}{\partial x^i}x'^g + \delta^{ik}\frac{\partial a^n}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \quad (43)$$

$$\Gamma'^{n}_{0l} = \Gamma'^{n}_{l0} = -2\frac{d\dot{a}^n}{dx^l} - 2\left(A_k\right)^n \left(\dot{A}_j\right)^k \frac{da^j}{dx'^l} - 2\left(A_k\right)^n \frac{d\left(A_j\right)^k}{dx'^l}\dot{a}^j - 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l}a^j$$

$$+2\left(A_k\right)^n \left(\dot{A}_l\right)^k + 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l}x'^j \quad (44)$$

These terms can be simplified, because:

$$-\frac{d^2a^n}{dx'^l dx'^m} - 2\left(A_k\right)^n \frac{d\left(A_j\right)^k}{dx'^l}\frac{da^j}{dx'^m} - \left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}a^j = \left(A_k\right)^n \frac{d^2}{dx'^l dx'^m}\left(\left(A_j\right)^k a^j\right) \quad (45)$$

$$-\ddot{a}^n - 2\left(A_k\right)^n \left(\dot{A}_j\right)^k \dot{a}^j - \left(A_k\right)^n \left(\ddot{A}_j\right)^k a^j = \left(A_k\right)^n \frac{d^2}{dt^2}\left(\left(A_j\right)^k a^j\right) \quad (46)$$

$$-2\frac{d\dot{a}^n}{dx^l} - 2\left(A_k\right)^n \left(\dot{A}_j\right)^k \frac{da^j}{dx'^l} - 2\left(A_k\right)^n \frac{d\left(A_j\right)^k}{dx'^l}\dot{a}^j - 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l}a^j = 2\left(A_k\right)^n \frac{d^2}{dtdx'^l}\left(\left(A_j\right)^k a^j\right) \quad (47)$$

These properties are true because $\frac{dt}{dx'^l} = 0$
Then we now have Coefficients:

$$\Gamma'^{n}_{lm} = \left(A_k\right)^n \frac{d^2}{dx'^l dx'^m}\left(\left(A_j\right)^k a^j\right) + \left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^l dx'^m}x'^j + 2\left(A_k\right)^n \frac{d\left(A_m\right)^k}{dx'^l} \quad (48)$$

$$\Gamma'^{n}_{0l} = \Gamma'^{n}_{l0} = 2\left(A_k\right)^n \frac{d^2}{dtdx'^l}\left(\left(A_j\right)^k a^j\right) + 2\left(A_k\right)^n \left(\dot{A}_l\right)^k + 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l}x'^j \quad (49)$$

$$\Gamma'^{n}_{00} = -\left(A_k\right)^n \frac{d^2}{dt^2}\left(\left(A_j\right)^k a^j\right) + \left(A_k\right)^n \left(\ddot{A}_j\right)^k x'^j$$

$$+\left(\delta^{in}\frac{\partial x'^k}{\partial x^i} + \delta^{ik}\left(A_p\right)^n \frac{\partial\left(A_g\right)^p}{\partial x^i}x'^g + \delta^{ik}\frac{\partial a^n}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \quad (50)$$

We must first show that these terms simplify to the autoparrallel coefficients from the Galilean Line Group in the previous paper under no spatial dependence. The coefficients from the previous paper [1] are:

$$\Gamma\text{'}^n_{lm} = 0 \ (51)$$

$$\Gamma\text{'}^n_{0l} = \Gamma\text{'}^n_{l0} = 2 \quad A^{nk}\dot{A}_{lk} \ (52)$$

$$\Gamma\text{'}^n_{00} = \delta^{kn}\frac{\partial\Phi}{\partial x'^k} + A^{nk}\ddot{A}_{jk}x'^j - A^{kn}\frac{d^2}{dt^2}\left(A_{jk}a^j\right) \ (53)$$

When $(A_j)^k$ and a^j are only functions of t:

$$\frac{d\,(A_j)^k}{dx'^q} = \frac{da^j}{dx'^q} = 0 \ (54),$$

where q is any nonzero index. Then it is clear (48) simplifies to (51), (49) simplifies to (52), and (50) simplifies to (53) under these conditions.

The symmetry of the $\Gamma\text{'}^n_{l0} = \Gamma\text{'}^n_{0l}$ is clear from the form of the terms of (49). Note, however, while the first two terms of the $\Gamma\text{'}^n_{lm}$ coefficient are clearly symmetric, as

$$(A_k)^n \frac{d^2}{dx'^l dx'^m}\left((A_j)^k a^j\right) = (A_k)^n \frac{d^2}{dx'^m dx'^l}\left((A_j)^k a^j\right) \ (50)$$

$$(A_k)^n \frac{d^2\,(A_j)^k}{dx'^l dx'^m}x'^j = (A_k)^n \frac{d^2\,(A_j)^k}{dx'^m dx'^l}x'^j \ (51),$$

The term

$$2\,(A_k)^n \frac{d\,(A_m)^k}{dx'^l} \neq 2\,(A_k)^n \frac{d\,(A_l)^k}{dx'^m} \ (52)$$

in general. Therefore, these coefficients are not independently symmetric. However, the autoparrallel equation (6) requires these terms to be symmetric when summed over in the equation, because:

$$\frac{dx'^l}{d\tau}\frac{dx'^m}{d\tau} = \frac{dx'^m}{d\tau}\frac{dx'^l}{d\tau} \ (53)$$

This is required by the communativity of standard multiplication of functions, and therefore, when summed over these derivatives, all three coefficient terms are required to be symmetric in the transformed autoparallel equation. However, note that this non-symmetric term within the coefficients of the transformed autoparallel equation (7b) will result in torsion within this thoery. Therefore, note that, even in the weak limit, this model cannot lead to the same results as General Relativity, as the manifold for GR has no torsion.

Finally, note that, though it was not done in this project because of our focus on the semi-group-properties and autoparallel equation, as a vector field the gravitational field $\vec{g}'$ can be written as the summation of the gradient of a scalar potential and the curl of a vector potential, as in [1] [16].

Ricci Tensor and Torsion Tensor

To calculate the Ricci tensor through the coefficients in the transformed autoparallel equation (48, 49, and 50), we use the definition of the Ricci tensor defined in [1]:

$$R_{\alpha\beta} = \frac{\partial}{\partial x^\rho}\Gamma^\rho_{\beta\alpha} - \frac{\partial}{\partial x^\beta}\Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\rho\alpha} \quad (54)$$

For the transformed R_{00} term, this becomes:

$$R_{00} = \frac{\partial}{\partial x'^n}\Gamma'^n_{00} - \frac{\partial}{\partial t'}\Gamma'^n_{n0} + \Gamma'^n_{nl}\Gamma'^l_{00} - \Gamma'^n_{0l}\Gamma'^l_{n0} \quad (55)$$

We calculated:

$$
\begin{aligned}
\frac{\partial}{\partial x^n}\Gamma'^{n}_{00} & \\
&= \frac{\partial}{\partial x^n}(A_k)^n \frac{d^2}{dt^2}\left[(A_j)^k a^j\right] + (A_j)^k \frac{\partial}{\partial x^n}\frac{d^2}{dt^2}\left[(A_j)^k a^j\right] + \frac{\partial}{\partial x^n}(A_k)^n \left(\ddot{A}_k\right)^k x'^j \\
&+ (A_k)^n \frac{\partial\left(\ddot{A}_j\right)^k}{\partial x^n} x'^j + (A_k)^n (A_j)^k \frac{\partial x'^j}{\partial x'^n} + \\
&+ \frac{\partial^2\Phi}{\partial x'^k \partial x'^n}\left(\delta^{ik}\frac{\partial x'^n}{\partial x^i} + \delta^{ik}(A_p)^n \frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}\frac{\partial a^n}{\partial x^i}\right) \\
&+ \frac{\partial\Phi}{\partial x'^k}(\delta^{ik}\frac{\partial (A_p)^n}{\partial x'^n}\frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}(A_p)^n \frac{\partial^2 (A_g)^p}{\partial x^i \partial x'^n} \\
&+ \delta^{ik}\delta^g_n (A_p)^n \frac{\partial (A_g)^p}{\partial x^i} + \delta^{ik}\frac{\partial^2 a^n}{\partial x^i \partial x'^n}) \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\Gamma'^{n}_{0l} &= 2\left(\dot{A}_k\right)^n \frac{d^2}{dtdx'^l}\left((A_j)^k a^j\right) + 2(A_k)^n \frac{d^3}{dt^2 dx'^l}\left((A_j)^k a^j\right) + 2\left(\dot{A}_k\right)^n \left(\dot{A}_l\right)^k \\
&+ 2(A_k)^n \left(\ddot{A}_l\right)^k + 2\left(\dot{A}_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l} x'^j + 2(A_k)^n \frac{d\left(\ddot{A}_j\right)^k}{dx'^l} x'^j + 2(A_k)^n \frac{\partial (A_j)^k}{\partial x'^l}\dot{x}^l \qquad (57)
\end{aligned}
$$

$$
\Gamma'^{n}_{nm} = (A_k)^n \frac{d^2}{dx'^n dx'^m}\left((A_j)^k a^j\right) + (A_k)^n \frac{d^2 (A_j)^k}{dx'^n dx'^m} x'^j + 2(A_k)^n \frac{d(A_m)^k}{dx'^n} \qquad (58)
$$

$$
\begin{aligned}
\Gamma'^{l}_{00} &= -(A_k)^l \frac{d^2}{dt^2}\left((A_j)^k a^j\right) + (A_k)^l \left(\ddot{A}_j\right)^k x'^j \\
&+ \left(\delta^{ik}\frac{\partial x'^l}{\partial x^i} + \delta^{ik}(A_p)^l \frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}\frac{\partial a^l}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma'^{n}_{nm}\Gamma'^{l}_{00} &= -(A_k)^n \frac{d^2}{dx'^n dx'^m}\left((A_j)^k a^j\right)(A_k)^l \frac{d^2}{dt^2}\left((A_j)^k a^j\right) \\
&+ +(A_k)^n \frac{d^2}{dx'^n dx'^m}\left((A_j)^k a^j\right)(A_k)^l \left(\ddot{A}_j\right)^k x'^j \\
&+ (A_k)^n \frac{d^2}{dx'^n dx'^m}\left((A_j)^k a^j\right)\left(\delta^{ik}\frac{\partial x'^l}{\partial x^i} + \delta^{ik}(A_p)^l \frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}\frac{\partial a^l}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \\
&+ -(A_k)^n \frac{d^2 (A_j)^k}{dx'^n dx'^m} x'^j (A_k)^l \frac{d^2}{dt^2}\left((A_j)^k a^j\right) + (A_k)^n \frac{d^2 (A_j)^k}{dx'^n dx'^m} x'^j (A_k)^l \left(\ddot{A}_j\right)^k x'^j \\
&+ +(A_k)^n \frac{d^2 (A_j)^k}{dx'^n dx'^m} x'^j \left(\delta^{ik}\frac{\partial x'^l}{\partial x^i} + \delta^{ik}(A_p)^l \frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}\frac{\partial a^l}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \\
&+ -2(A_k)^n \frac{d(A_m)^k}{dx'^n}(A_k)^l \frac{d^2}{dt^2}\left((A_j)^k a^j\right) + 2(A_k)^n \frac{d(A_m)^k}{dx'^n}(A_k)^l \left(\ddot{A}_j\right)^k x'^j \\
&+ 2(A_k)^n \frac{d(A_m)^k}{dx'^n}\left(\delta^{ik}\frac{\partial x'^l}{\partial x^i} + \delta^{ik}(A_p)^l \frac{\partial (A_g)^p}{\partial x^i} x'^g + \delta^{ik}\frac{\partial a^l}{\partial x^i}\right)\frac{\partial\Phi}{\partial x'^k} \qquad (60)
\end{aligned}
$$

$$
\Gamma'^{n}_{0l} = 2(A_k)^n \frac{d^2}{dtdx'^l}\left((A_j)^k a^j\right) + 2(A_k)^n \left(\dot{A}_l\right)^k + 2(A_k)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l} x'^j \qquad (61)
$$

$$
\Gamma'^{l}_{n0} = 2(A_k)^l \frac{d^2}{dtdx'^n}\left((A_j)^k a^j\right) + 2(A_k)^l \left(\dot{A}_n\right)^k + 2(A_k)^l \frac{d\left(\dot{A}_j\right)^k}{dx'^n} x'^j \qquad (62)
$$

$$
\begin{aligned}
\Gamma'^{n}_{0l}\Gamma'^{l}_{n0} &= 2\left(A_k\right)^n \frac{d^2}{dtdx'^l}\left(\left(A_j\right)^k a^j\right) 2\left(A_k\right)^l \frac{d^2}{dtdx'^n}\left(\left(A_j\right)^k a^j\right) \\
&+ 2\left(A_k\right)^n \frac{d^2}{dtdx'^l}\left(\left(A_j\right)^k a^j\right) 2\left(A_k\right)^l \left(\dot{A}_n\right)^k \\
&+ 2\left(A_k\right)^n \frac{d^2}{dtdx'^l}\left(\left(A_j\right)^k a^j\right) 2\left(A_k\right)^l \frac{d\left(\dot{A}_j\right)^k}{dx'^n} x'^j + 2\left(A_k\right)^n \left(\dot{A}_l\right)^k 2\left(A_k\right)^l \frac{d^2}{dtdx'^n}\left(\left(A_j\right)^k a^j\right) \\
&+ 2\left(A_k\right)^n \left(\dot{A}_l\right)^k 2\left(A_k\right)^l \left(\dot{A}_n\right)^k + 2\left(A_k\right)^n \left(\dot{A}_l\right)^k 2\left(A_k\right)^l \frac{d\left(\dot{A}_j\right)^k}{dx'^n} x'^j \\
&+ 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l} x'^j 2\left(A_k\right)^l \frac{d^2}{dtdx'^n}\left(\left(A_j\right)^k a^j\right) + 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l} x'^j 2\left(A_k\right)^l \left(\dot{A}_n\right)^k \\
&+ 2\left(A_k\right)^n \frac{d\left(\dot{A}_j\right)^k}{dx'^l} x'^j 2\left(A_k\right)^l \frac{d\left(\dot{A}_j\right)^k}{dx'^n} x'^j \quad (63)
\end{aligned}
$$

While in the process of simplifying this, we also sought to examine the R'_{lm}, defined as:

$$
R'_{lm} = \frac{\partial}{\partial x'^n}\Gamma'^{n}_{ml} - \frac{\partial}{\partial x'^m}\Gamma'^{n}_{nl} + \Gamma'^{n}_{ns}\Gamma'^{s}_{ml} - \Gamma'^{n}_{ms}\Gamma'^{s}_{nl} \quad (64)
$$

We found that the first terms become:

$$
\begin{aligned}
\frac{\partial}{\partial x'^n}\Gamma'^{n}_{ml} - \frac{\partial}{\partial x'^m}\Gamma'^{n}_{nl} &= \frac{\partial\left(A^n_p\right)}{\partial x'^n}\frac{d^2}{dx'^m dx'^l}\left[\left(A_q\right)^p a^q\right] - \frac{\partial\left(A_p\right)^n}{\partial x'^m}\frac{d^2}{dx'^l dx'^n}\left[\left(A_q\right)^p a^q\right] \\
&+ \frac{\partial\left(A_q\right)^n}{\partial x'^m}\frac{d^2\left(A_q\right)^p}{dx'^m dx'^l}x'^q + \left(A_p\right)^n \frac{d^2\left(A_q\right)^p}{dx'^m dx'^l}\delta^q_n - \frac{\partial\left(A_p\right)^n}{\partial x'^m}\frac{d^2\left(A_q\right)^p}{dx'^l dx'^n}x'^q \\
&- \left(A_p\right)^n \frac{d^3\left(A_q\right)^p}{dx'^l dx'^n}\delta^q_m + 2\frac{\partial\left(A_p\right)^n}{\partial x'^n}\frac{d\left(A_l\right)^p}{dx'^n} \quad (65) \\
\Gamma'^{n}_{ns}\Gamma'^{s}_{ml} &= \left(\left(A_k\right)^n \frac{d^2}{dx'^n dx'^s}\left[\left(A_j\right)^k a^j\right] + \left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^n dx'^s}x'^j + 2\left(A_k\right)^n \frac{d\left(A_s\right)^k}{dx'^n}\right) \\
&\times \left(\left(A_k\right)^s \frac{d^2}{dx'^m dx'^l}\left[\left(A_j\right)^k a^j\right] + \left(A_k\right)^s \frac{d^2\left(A_j\right)^k}{dx'^m dx'^l}x'^j + 2\left(A_k\right)^s \frac{d\left(A_m\right)^k}{dx'^l}\right) \quad (66) \\
\Gamma'^{n}_{ms}\Gamma'^{s}_{nl} &= \left(\left(A_k\right)^n \frac{d^2}{dx'^m dx'^s}\left[\left(A_j\right)^k a^j\right] + \left(A_k\right)^n \frac{d^2\left(A_j\right)^k}{dx'^m dx'^s}x'^j + 2\left(A_k\right)^n \frac{d\left(A_s\right)^k}{dx'^m}\right) \\
&\times \left(\left(A_k\right)^s \frac{d^2}{dx'^n dx'^l}\left[\left(A_j\right)^k a^j\right] + \left(A_k\right)^s \frac{d^2\left(A_j\right)^k}{dx'^n dx'^l}x'^j + 2\left(A_k\right)^s \frac{d\left(A_l\right)^k}{dx'^n}\right) \quad (67)
\end{aligned}
$$

Subtracting (66) from (65) must yield the terms:

$$4\left(A_k\right)^n \frac{d\left(A_s\right)^k}{dx\text{'}^n}\left(A_k\right)^s \frac{d\left(A_m\right)^k}{dx\text{'}^l} - 4\left(A_k\right)^n \frac{d\left(A_s\right)^k}{dx\text{'}^m}\left(A_k\right)^s \frac{d\left(A_l\right)^k}{dx\text{'}^n} \ (68)$$

which does not appear to be zero, given the different indeces and lack of symmetry for these terms within the $\Gamma\text{'}^n_{ml}$ coefficients. However, we know, as a tensor, the Ricci tensor must be invariant, and thus another way to derive them is to transform them as follows:

$$R\text{'}_{ml} = \frac{\partial t}{\partial x\text{'}^m}\frac{\partial t}{\partial x\text{'}^l}R_{00} \ (69)$$

We know that $t\text{'} = t + b$ as defined in (22), and therefore:

$$\frac{\partial t}{\partial x\text{'}^d} = 0 \ (70)$$

for any nonzero d. Then for consistency it must be the case that either (68) reduces to zero, or there is another simplification involving these single-derivative terms.

We can conclude similarly to (68) that $R\text{'}_{0l} = R\text{'}_{l0} = 0$, as both will have a term as in (70).

We can also conclude further:

$$R\text{'}_{00} = \frac{\partial t}{\partial t\text{'}}\frac{\partial t}{\partial t\text{'}} = R_{00} = 4\pi\rho \ (71)$$

and therefore (55) with the substitutions of (56), (57), (60), and (63) also simplifies to $4\pi\rho$ as in [1]. With further research, we could conclude the implications of this and prove this fully.

We derive the torsion tensor, defined in [15] and [16] as:

$$\nabla_{\bar{U}}\bar{V} - \nabla_{\bar{V}}\bar{U} - \left[\bar{U},\bar{V}\right] \ (72)$$

Given our coordinate choice and the communitivity of partial derivatives, this equation reduces to:

$$\Gamma\text{'}^n_{lm} - \Gamma\text{'}^n_{ml} = 2\left(A_k\right)^n \frac{d\left(A_m\right)^k}{dx'^l} - 2\left(A_k\right)^n \frac{d\left(A_l\right)^k}{dx'^m} \ (73)$$

This implies a nonzero torsion tensor, and a deviation from General Relativity in all cases, including the weak field limit, as General Relativity does not have torsion. For a further examination of the weak field limit of general relavity, see [14,15].

Closing Remarks

In this paper, we have studied the semi-group designated $\mathbb{G}*$, which is an extension of the Galilean Line group $\mathbb{G}$ from [1]. We established that, as a semi-group, $\mathbb{G}*$ has an identity, a is associative, and is closed, but a general element does not have an inverse. We prove the lack of an inverse for one form of elements of $\mathbb{G}*$. We noted that all the coefficients of the auto-parallel eqaution, $\Gamma\text{'}^{n}_{00}$, $\Gamma\text{'}^{n}_{l0}$, and $\Gamma\text{'}^{n}_{lm}$, are nonzero in the general case in $\mathbb{G}*$. Upon examination of the Ricci tensor of $\mathbb{G}*$, we notice an inconsistency between two separate derivations of the Ricci tensor. We define the nonzero torsion tensor for the transformed field.

In the future, we would like to determine the smallest group containing $\mathbb{G}*$which also contains the inverses of $\mathbb{G}*$. We know the general group of diffeomorphisms contains $\mathbb{G}*$ and its inverses, but wish to determine if there is a non-trivial subgroup of the group of diffemorphisms containing both $\mathbb{G}*$ and its inverses. We would also like to examine the properties of the Γ' coefficient in order to attempt to find a physical analogue to the terms, as in [1]. We would like to determine a general form for the gravitational field $\vec{g}$ as a sum of the gradient of a scalar potential and the curl of a vector potential. Finally, we are interested in a different extension of the Galilean Line Group in which t' is a spatially dependent, and thus we lose absolute time. What properties would this transformation set reveal? We conclude that though we found $\mathbb{G}*$ untenable, the results of this study bring light to the uniqueness of the Galilean Line Group and the consistency of it and general relativity.

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