Introduction

LU decomposition factors a matrix into the product of a lower-triangular matrix (L) and an upper-triangular matrix (U). The goal of this research is to use LU decomposition to build new identities for some special matrices. We study matrices that arise in the study of three-term recurrence relations, orthogonal polynomials, graph theory, Euler numbers, Bernoulli numbers, and discrete Fourier transforms. We use Maple (a symbolic computational software) and OEIS (Online Encyclopedia of Integer Sequence) to assist our computations and sequence identification.

Our approach follows the methodology used in Prof. Chamberland’s paper:
- perform LU decomposition on a matrix
- identify the patterns of the resulting L and U matrices and make conjectures
- build new identities for the original matrix
- make possible interpretations
- write up formal proofs if possible

Results

1. General LU Decomposition

Knowing both L and U enables us to reconstruct the original matrix A using the formula: $A = \sum_{i=1}^{n} L_i U_i$

In most cases, the form of this decomposition isn’t very practical computationally, but it can lead to interesting identities concerning the entries of the L and U matrices. This representation is built upon Householder’s work.

2. Three-term recurrence relations

Suppose we have a three-term recurrence relation: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ where $c_1$ and $c_2$ are constants, with initial conditions $a_0$ and $a_1$. We consider the associated Hankel matrices from the theory of orthogonal polynomials.

Applying LU decomposition to the matrix A gives a L of only first two columns having nonzero entries and the diagonal of L is replaced by j-th row in certain minors of the original matrix. Let $A_i(k,l)$ be the principal minor of A composed from the first k rows and columns. Let $N_i(k)$, the submatrix of A where the k-th column of A is replaced by the i-th column. Similarly the k-th row of A is replaced by j-th row of $A_j(k,l)$. Thus the following is true: $A_i(k,l) = \begin{bmatrix} A_i(k,l) & N_i(k) \\ N_i(k) & A_j(k,l) \end{bmatrix}$

In the case of the three-term recurrence relations, it is not hard to prove the following identity:

$A_i(k,l) = \begin{bmatrix} A_i(k,l) & N_i(k) \\ N_i(k) & A_j(k,l) \end{bmatrix}$

3. Orthogonal polynomials

In Legendre Polynomials

Orthogonal polynomials, which also satisfy three-term recurrence relations, produce lovely formula under LU decomposition. Legendre polynomial is one of well-known orthogonal polynomials involving Rodrigues’s formula: $P_n(x) = (n+1) \frac{d^n}{dx^n} \left( (1-x^2) \right)^{n+1}$. We construct a matrix composed on the Rodrigues’s formula by letting the (i, j) entry to be:

$L_{ij} = \begin{cases} \frac{1}{i+1} & \text{if } i = j \\
\frac{(-1)^{i-j}}{i+j} & \text{if } i < j \\
0 & \text{otherwise} \end{cases}$

We also have the Fibonacci sequence associated Hankel matrices with Chebyshev Polynomials.

4. Example with Vandermonde matrix

In Vandermonde matrix the i-th row consists of the consecutive powers of single variable $x_i$. The LU decomposition was originally performed by OrecHalil and yielded the following terms: $V = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$

The LU decomposition produces the following formula: $L_{ij} = \begin{bmatrix} \sum_{k=1}^{n} x_{i+k} \cdot x_{j+k} \\ \frac{1}{2} \sum_{k=1}^{n} x_{i+k} \cdot x_{j+k} \end{bmatrix}$

5. Euler and Bernoulli matrices

Euler matrix is a matrix $M$ such that $M \cdot \binom{n}{2}$ satisfies the matrix $A$ with entries $1$ and $1$, with $\binom{2}{2}$.

For example, when $n = 2$, the $M$ matrix and its LU decomposition are:

<table>
<thead>
<tr>
<th>$L_{ij}$</th>
<th>$U_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Applying this identity for the triangular terms of the matrix, we get:

$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} = \binom{n}{2}$

Similarly, Bernoulli matrix is a matrix $N$ such that $N \cdot \binom{n}{2}$ satisfies the matrix $A$ with entries $1$ and $1$, with $\binom{2}{2}$.

For example, when $n = 3$, the $N$ matrix and its LU decomposition are:

<table>
<thead>
<tr>
<th>$L_{ij}$</th>
<th>$U_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

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References


Bernoulli Numbers: Summation (formula 04.12.23.0001)." Online Encyclopedia of Integer Sequence (OEIS), 2015.


Euler numbers: Summation (formula 04.12.23.0001)." Online Encyclopedia of Integer Sequence (OEIS), 2015.


